INTRODUCTION TO THE ALGEBRAIC GEOMETRY OF A PLANE

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INTRODUCTION TO THE ALGEBRAIC GEOMETRY OF A PLANE

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PREFACE

I hope that this book will be of use to university undergraduates and to advanced students in schools, who have already learned to study simple properties of lines and circles by means of co-ordinates.

The text covers the usual course for an honours degree and goes some way further in preparation for a study of the general theory of algebraic curves.

The initial developments of the theory are presented in a manner which may appeal to students who require a careful account of the basic ideas in elementary algebraic geometry but who are unable to find time for, or wish to defer, a critical study of the axiomatic foundations of geometry. Those who are interested in a more sophisticated approach will naturally consult H F Baker's *Principles of Geometry* and his later *Plane Geometry*, Veblen and Young's *Projective Geometry*, and Veblen and Whitehead's tract on the *Foundations of Differential Geometry*.

This book starts from the familiar ideas of a real euclidean plane and describes the modifications which must be made so that the algebra of complex numbers and of homogeneous polynomials may be used in a logical manner. The main subject matter is to be summed up as a study in projective transformations, especially of lines and conics, and in the geometrical and numerical invariants relating to these transformations Metrical geometry is treated as a branch of the projective theory in which special significance is attached to two particular points The last chapter introduces the reader, through the theory of rational curves, to methods and ideas which are used in the general theory of algebraic curves.

Certain simple properties of equations, determinants and matrices are assumed as occasion arises, the reader will normally be familiar with these by the stages at which they are reached. but he would do well to read as early as possible the elements of those branches of algebra.

A very few new terms and slight modifications of old terms have been introduced; I hope that they may be acceptable

PREFACE

I wish to express my gratitude to those who have assisted me in the preparation of this book. I make this an opportunity to acknowledge the inspiration of my own teachers in the past and the helpful discussions at various times with my colleagues, particularly Mr. T. L. Wren. I am especially grateful to Mr. Wren and Mr. H. Kestelman, who have read the book in manuscript and in proof and to whom are due a number of valuable improvements in the text. Through the kindness of the University of London, I have been able to include at the end of the book a collection of problems set by the University in recent years.

J. W ARCHBOLD.

London,
September 1947.

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CHAPTER I

FOUNDATIONS

1. Introduction. The real euclidean plane E_R.

(i) The approach to the subject.—In presenting a theory of algebraic geometry we have to choose for our starting point between a formulation of geometry which is non-algebraic in character—the so-called pure geometry—and one which is wholly algebraic.

Here we are guided by the normal attainments and outlook of the readers for whom the book is intended. We therefore base the theory on a knowledge of the pure geometry of a euclidean plane, there being some advantage in starting within the boundaries of the readers' present knowledge.

It is reasonable to assume that those who are beginning to make a serious study of algebraic geometry have already a background of experience in the elementary treatment of the geometry of lines and circles by means of co-ordinates. Accordingly, this first section deals with some particular aspects of this background and goes into more than usual detail on some matters which are frequently left to intuition.

(ii) One-one correspondences and co-ordinates.—A fundamental mathematical concept is that of a *one-one*, or (1, 1), *correspondence* between two sets of objects. This is a rule which associates with every object of either set one and only one object of the other set.

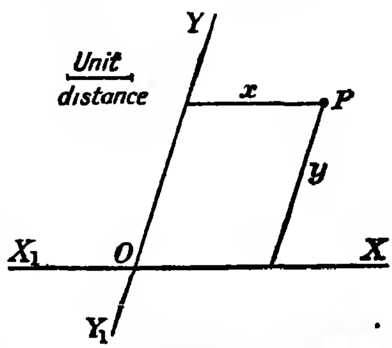


FIG. 1.—DISTANCE-CO-ORDINATES.

The basis of the algebraic geometry of a euclidean plane is the fact that it is possible, in many ways, to set up a (1, 1) correspondence between the points of the plane and the ordered pairs of real numbers. As is familiar (see Fig. 1), this may be done by selecting arbitrarily two intersecting lines X_1OX, Y_1OY, called the *axes of co-ordinates*, and some standard length as unit of distance; then we associate with every point P the ordered pair of numbers x, y, where x is the number of units of distance of P

from Y_1OY, measured parallel to X_1OX, and is positive or negative according as P, X do or do not lie on the same side of Y_1OY, and y is similarly defined The numbers x, y are called the *distance co-ordinates* of P relative to the selected axes and scale of measurement. In virtue of the (1, 1) correspondence, we refer to P as the point (x, y), bearing in mind to which axes and scale the two numbers refer

A rotation in the plane about the *origin* O in the sense in

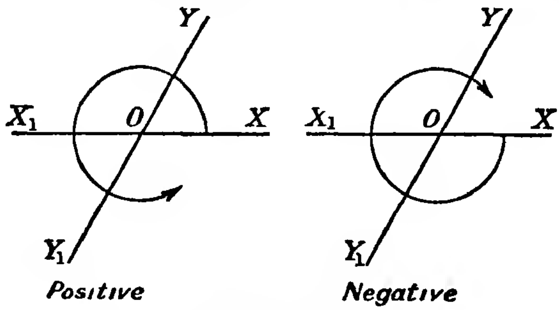


FIG 2.—SENSES OF ROTATION

which a half-line OP would occupy in turn the positions OX, OY is called *positive*; a rotation in the opposite sense is called *negative* (Fig 2).

(iii) The distance between two points.—Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of two points P_1, P_2 and let the line through P_1 parallel to the axis X_1OX meet the line through P_2 parallel to the axis Y_1OY at Q (Fig. 3); then Q is the point (x_2, y_1)

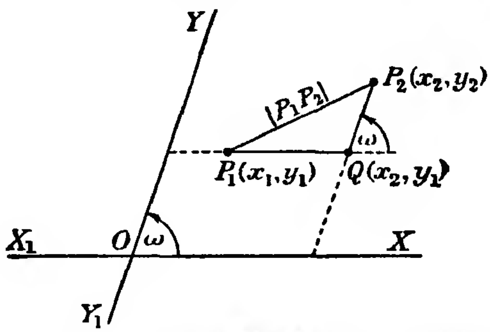


FIG 3.—DISTANCE BETWEEN TWO POINTS.

Denoting by $|P_1P_2|$, or equally by $|P_2P_1|$, the number of units of distance between P_1, P_2, and similarly for any other pair of points, we have:

$$|P_1Q| = \epsilon(x_1 - x_2),\ |P_2Q| = \epsilon'(y_1 - y_2),$$

where each of ϵ, ϵ' is either $+1$ or -1.

By the extended form of Pythagoras' theorem,

$$|P_1P_2|^2 = |P_1Q|^2 + |P_2Q|^2 + 2\epsilon|P_1Q| \cdot \epsilon'|P_2Q| \cos \omega,$$

where ω is the least angle such that a positive rotation through ω brings the half-line OX to the half-line OY; ω is called *the angle between the axes.*

Hence

$$|P_1P_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2) \cos \omega.$$

If $\omega = \pi/2$, the axes are said to be *rectangular*, the distance formula just given then takes the simpler form

$$|P_1P_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

It is chiefly on account of this simplification that rectangular axes are generally much more convenient in practice than are *oblique* (non-rectangular) axes

It should be remarked that $|P_1P_2|$ is a non-negative number of units of distance but is not itself an actual distance. Nevertheless, for brevity, we often speak of $|P_1P_2|$ as the distance between P_1, P_2; there is no ambiguity, since the abbreviation is used only in circumstances where a single scale for measuring distances is employed

(iv) Sensed distances.—(*a*) It is very convenient in euclidean geometry to refine the notion of distance so as to take account of the order in which points lie on a line. Intuitively the basis of the refinement is simple to appreciate, and is expressible by the view that a point may move along a line AB in the sense from A towards B or in the sense from B towards A We do not here enter into full detail, but assume that the idea of one point lying between two others is understood On this matter the reader is recommended to consult H. F. Baker's *Principles of Geometry*, Vol I, Ch. II

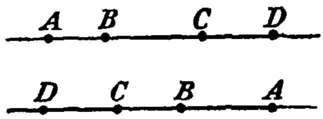


FIG. 4 —SUCCESSIVE POINTS, A, B, C, D.

If A, B, C, D are four points in line such that B lies between A and C and C lies between B and D, we say that these points are *successive* in the order A, B, C, D or in the order D, C, B, A (Fig 4)

Let P, Q be any two points on the line AB. We define as follows a number $\overrightarrow{PQ}_{AB}$, called the *distance from* P *to* Q *in the sense from* A *to* B. If the four points are successive in any of the orders P, Q, A, B, P, A, Q, B; A, P, Q, B; A, P, B, Q, A, B, P, Q,

we take $\overrightarrow{PQ}_{AB}$ to be $+|PQ|$. If the four points are not successive in any of these orders, we take $\overrightarrow{PQ}_{AB}$ to be $-|PQ|$.

Intuitively, we have said that $\overrightarrow{PQ}_{AB} = +|PQ|$ if a point moves from P towards Q in the same direction as from A towards B, and that $\overrightarrow{PQ}_{AB} = -|PQ|$ if a point moves from P towards Q in the opposite direction to that from A towards B (Fig. 5).

$\overrightarrow{PQ}_{AB} = +|PQ|$ $\qquad\qquad$ $\overrightarrow{PQ}_{AB} = -|PQ|$

FIG 5.—SENSED DISTANCE.

Ex 1.
$$\overrightarrow{PQ}_{AB} = -\overrightarrow{QP}_{AB}.$$
$$\overrightarrow{AB}_{AB} = -\overrightarrow{BA}_{AB} = |AB|.$$

Ex. 2. If P, Q, R are points on the line AB, then
$$\overrightarrow{PQ}_{AB} + \overrightarrow{QR}_{AB} = \overrightarrow{PR}_{AB},$$
which is equivalent to
$$\overrightarrow{PQ}_{AB} + \overrightarrow{QR}_{AB} + \overrightarrow{RP}_{AB} = 0.$$

Ex. 3. If P is the point $(x, 0)$ on X_1OX, then
$$\overrightarrow{OP}_{OX} = x;$$
and if Q is the point $(x', 0)$, then
$$\overrightarrow{PQ}_{OX} = x' - x.$$

(*b*) It is frequently undesirable to refer explicitly to the points A, B which serve to determine a sense on the line. We then use the notation $\overrightarrow{PQ}$ in place of $\overrightarrow{PQ}_{AB}$, calling $\overrightarrow{PQ}$ a *sensed distance*; it should, however, be borne in mind that this convention implies a preliminary choice of two points such as A, B: generally the context indicates such a pair For example, in the case of points on the axis X_1OX one would naturally mean $\overrightarrow{PQ}_{OX}$ when writing $\overrightarrow{PQ}$.

Ex. 4. If P, Q, R, S are collinear points, prove that
$$\overrightarrow{QR}\,.\,\overrightarrow{PS} + \overrightarrow{RP}\,.\,\overrightarrow{QS} + \overrightarrow{PQ}\,.\,\overrightarrow{RS} = 0.$$

(v) Joachimstal's section-formulae.—Let P be a point on the line AB and let $\overrightarrow{AP} : \overrightarrow{PB} = \lambda : \mu$. P is said to divide $\overrightarrow{AB}$ in the ratio $\lambda : \mu$. There is no position of P for which $\overrightarrow{AP} = \overrightarrow{BP}$; therefore $\lambda + \mu \neq 0$.

Joachimstal's formulae express the co-ordinates (x, y) of P in terms of λ, μ and the co-ordinates (x_1, y_1) of A and (x_2, y_2) of B; they are (with axes oblique or rectangular)

$$x = \frac{\lambda x_2 + \mu x_1}{\lambda + \mu}, \quad y = \frac{\lambda y_2 + \mu y_1}{\lambda + \mu}.$$

The proof is trivial if AB is parallel to either axis of co-ordinates and is left to the reader. Let us suppose then (as in Fig. 6) that

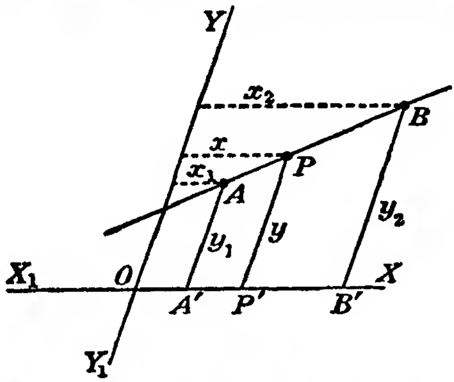


FIG. 6.—FIGURE FOR JOACHIMSTAL'S SECTION-FORMULAE.

AB is not parallel to either axis and let the parallels to Y_1OY through A, P, B meet X_1OX at A', P', B' respectively. Then also $\overrightarrow{A'P'} : \overrightarrow{P'B'} = \lambda : \mu$. Since $\overrightarrow{A'P'} = x - x_1$, $\overrightarrow{P'B'} = x_2 - x$, we have

$$x - x_1 : x_2 - x = \lambda : \mu,$$

giving

$$\mu(x - x_1) = \lambda(x_2 - x)$$

or

$$x = \frac{\lambda x_2 + \mu x_1}{\lambda + \mu}.$$

Similarly

$$y = \frac{\lambda y_2 + \mu y_1}{\lambda + \mu}.$$

Putting $\theta = \lambda/(\lambda + \mu)$, $\phi = \mu/(\lambda + \mu)$, so that $\theta + \phi = 1$, Joachimstal's formulae take the form, sometimes more convenient,

$$x = \theta x_2 + \phi x_1, \; y = \theta y_2 + \phi y_1, \; 1 = \theta + \phi.$$

(vi) The equation of a line.—We prove that, whatever may be the angle between the axes, the co-ordinates of the points on any given line are those and only those which satisfy an equation of the form

$$lx + my + n = 0,$$

where l, m, n are real numbers with l, m not both zero.

Let us select two distinct points on the line and let the co-ordinates of these be (x_1, y_1), (x_2, y_2). Let (x, y) be any point on the line. By the preceding part of this section there are numbers θ, ϕ such that

$$\begin{aligned} x &= \theta x_2 + \phi x_1, \\ y &= \theta y_2 + \phi y_1, \\ 1 &= \theta + \phi \end{aligned}$$

From the first pair of these equations in θ, ϕ, we have:

$$\theta : \phi : 1 = xy_1 - x_1y : x_2y - xy_2 : x_2y_1 - x_1y_2$$

and therefore, substituting in the last equation,

$$(xy_1 - x_1y) + (x_2y - xy_2) = x_2y_1 - x_1y_2.$$

Thus the co-ordinates of every point on the line satisfy the equation

$$x(y_1 - y_2) + y(x_2 - x_1) + (x_1y_2 - x_2y_1) = 0$$

which is of the required form; for the simultaneous vanishing of $y_1 - y_2$ and $x_2 - x_1$ would imply the identity of the point (x_1, y_1) with the point (x_2, y_2)

The left-hand side of the equation last written is usually expressed in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

This expression is called a *determinant*. Those who are as yet unfamiliar with determinants should be content for the moment to regard the determinant as synonymous with the left-hand side in question and to note that the elements in the first two columns are just the co-ordinates of the points (x, y), (x_1, y_1), (x_2, y_2). Those who are familiar with determinants will already have observed that the vanishing of this determinant may at once be inferred from the three equations involving θ, ϕ.

Conversely, let us consider the set of points whose co-ordinates (x, y) satisfy an equation $lx + my + n = 0$, with l, m not both

zero. There is clearly an infinity of points in the set; let two of these be (x_1, y_1), (x_2, y_2). Then

$$lx_1 + my_1 + n = 0,$$
$$lx_2 + my_2 + n = 0$$

and therefore

$$l : m : n = y_1 - y_2 : x_2 - x_1 : x_1y_2 - x_2y_1.$$

Thus the points of the set are those which satisfy the equation

$$x(y_1 - y_2) + y(x_2 - x_1) + (x_1y_2 - x_2y_1) = 0$$

and are therefore just those which belong to the line joining the points (x_1, y_1), (x_2, y_2).

The equation $lx + my + n = 0$ is called the *equation of the line*. It is of fundamental importance that it is *linear* in x, y, that is to say of the first degree in x, y.

Ex. 5. The equation of the line joining the origin to the point (x_1, y_1) is $xy_1 - x_1y = 0$. If $x_1y_1 \neq 0$, this equation is conveniently remembered in the form $x/x_1 = y/y_1$.

Ex. 6 The equation of the line joining the points $(a, 0)$, $(0, b)$, where $ab \neq 0$, is $x/a + y/b = 1$.

(vii) Co-ordinates of a line. Tersets.—The equation $lx + my + n = 0$, with l, m not both zero, represents the same line L as does the equation $\lambda lx + \lambda my + \lambda n = 0$, where λ is any real number except zero Any ordered set λl, λm, λn is called a set of *distance co-ordinates* of L relative to the axes of co-ordinates already selected In this sytem of co-ordinates we refer to L as the line (l, m, n) or as the line $(\lambda l, \lambda m, \lambda n)$ for any real value of λ except 0.

The word "distance" is used in the term "distance co-ordinates" simply to indicate that these co-ordinates are associated with the system of distance co-ordinates which have been used to determine the positions of points

The aggregate of ordered triads of numbers $(\lambda a, \lambda b, \lambda c)$, where a, b, c are fixed real numbers not all zero, and λ takes all real values except zero, is called a *real terset* and is denoted by $[a, b, c]$. Clearly $[a, b, c]$ and $[ka, kb, kc]$ denote the same terset, for all $k \neq 0$. The essential feature of a terset is that it depends on the ratios but not on the actual values of a, b, c.

A terset $[a, b, c]$ for which a, b are not both zero is here called *affine*.

The importance of the notion of terset lies at present in the fact, which is evident, that the lines L in a plane are in (1, 1) correspondence with the real affine tersets $[l, m, n]$.

(viii) Distance of a point from a line.—We suppose here that the axes of co-ordinates are rectangular.

We seek a formula for the shortest distance between a given point P (x_0, y_0) and the points of a line L (l, m, n); this distance is called the *distance of P from* L (Fig. 7.).

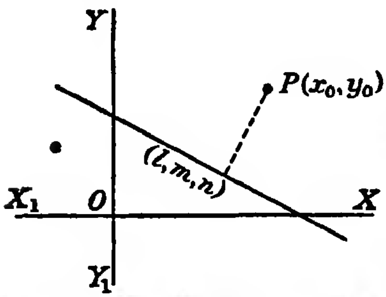


FIG. 7.—DISTANCE OF A POINT FROM A LINE.

In effect, we have to find the minimum value of the function of x, y

$$(x - x_0)^2 + (y - y_0)^2$$

subject to

$$lx + my + n = 0.$$

At least one of l, m is not zero; since these numbers are real, an equivalent statement is that $l^2 + m^2 \neq 0$. Let us suppose that $l \neq 0$; the argument will be similar if $l = 0, m \neq 0$. Then

$$x = -(my + n)/l$$

and therefore

$$\begin{aligned}
&(x - x_0)^2 + (y - y_0)^2 \\
&= \{(lx_0 + my + n)^2 + (ly - ly_0)^2\}/l^2 \\
&= \{(l^2 + m^2)y^2 + 2y(m(lx_0 + n) - l^2y_0) + (lx_0 + n)^2 + l^2{y_0}^2\}/l^2 \\
&= \frac{l^2 + m^2}{l^2}\left\{y + \frac{m(lx_0 + n) - l^2y_0}{l^2 + m^2}\right\}^2 \\
&\qquad + \frac{1}{l^2}\left\{(lx_0 + n)^2 + l^2{y_0}^2 - \frac{(m(lx_0 + n) - l^2y_0)^2}{l^2 + m^2}\right\} \\
&= \frac{l^2 + m^2}{l^2}\left\{y + \frac{m(lx_0 + n) - l^2y_0}{l^2 + m^2}\right\}^2 + \frac{(lx_0 + my_0 + n)^2}{l^2 + m^2}.
\end{aligned}$$

The minimum value of $(x - x_0)^2 + (y - y_0)^2$ is thus

$$\frac{(lx_0 + my_0 + n)^2}{l^2 + m^2}$$

and this number is the square of the distance sought.

The analysis just given is not the shortest possible, but it lends itself to further considerations which will be mentioned in *Ex.* 8 and in *Ex.* 10 below.

The number

$$\frac{lx_0 + my_0 + n}{\sqrt{l^2 + m^2}},$$

where the positive square root is meant (throughout the book) by the symbol $\sqrt{\ }$, is called the *algebraic distance* of the point (x_0, y_0)

from the line (l, m, n). Regarded as a function of x_0, y_0 it is continuous, and vanishes only for points lying on the line. The points of the plane which are not on the line thus fall algebraically into two sets: those on one side of the line have a positive algebraic distance from the line and those on the other side have a negative algebraic distance from it. Which side is "positive" and which "negative" evidently depends on which triad of the terset $[l, m, n]$ is used to name the line.

Ex. 7. The co-ordinates (x, y) of a point are the algebraic distances of the point from the lines $(1, 0, 0)$, $(0, 1, 0)$ respectively.

Ex. 8. The point on the line (l, m, n) which is nearest to the point (x_0, y_0) is

$$\left(\frac{m^2x_0 - l(my_0 + n)}{l^2 + m^2}, \frac{l^2y_0 - m(lx_0 + n)}{l^2 + m^2}\right).$$

(ix) Parallel lines. Principal angles.—Relative to oblique or rectangular axes, the equations

$$l_1x + m_1y + n_1 = 0,$$
$$l_2x + m_2y + n_2 = 0$$

represent different lines if and only if

$$l_1 : m_1 : n_1 \neq l_2 : m_2 : n_2.$$

When this inequality is satisfied, the two lines meet in the point given by

$$x : y : 1 = m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1$$

except and only except when $l_1m_2 - l_2m_1 = 0$, in which case the lines have no point in common. Thus a necessary and sufficient condition for the two lines to be *parallel* is

$$l_1m_2 - l_2m_1 = 0$$

or

$$l_1 : m_1 = l_2 : m_2.$$

Now let us suppose the axes to be rectangular and consider any line L (l, m, n) not parallel to either axis. The parallel line L' through the origin has the equation $lx + my = 0$, by what has just been said.

Let $P(x_1, y_1)$ be a point on L' for which $y_1 > 0$. Let θ be the least angle such that a positive rotation through θ brings the half line OX to the position of the half-line OP; then θ is called the *principal angle from* X_1OX *to* L (Fig 8). Any angle $\theta + k\pi$, where k is any integer, positive or negative, is called *an angle* from X_1OX to L. Evidently $0 < \theta < \pi$.

We have

$$x_1 : y_1 = m : -l,$$

together with

$$y_1 > 0.$$

Hence θ is determined by the equations

$$\cos\theta = \frac{\epsilon m}{\sqrt{l^2+m^2}},\ \sin\theta = \frac{-\epsilon l}{\sqrt{l^2+m^2}},$$

where ϵ is assigned that value $+1$ or -1 which makes $-\epsilon l$ positive.

Further, we have

$$\tan\theta = -l/m.$$

We call the number $-l/m$ the *gradient* of L.

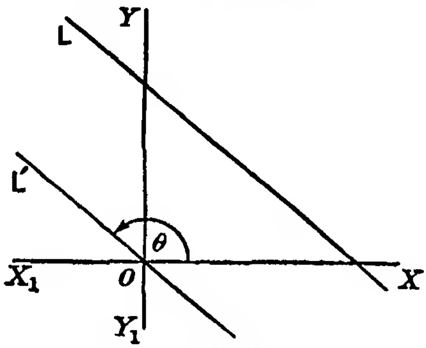


FIG 8 —PRINCIPAL ANGLE.

In the case of a line parallel to X_1OX, we define the principal angle to be 0; and in the case of a line parallel to Y_1OY to be $\pi/2$. These principal angles are given by the above formulae for $\cos\theta$, $\sin\theta$ provided that in the first case ϵ is assigned that value $+1$ or -1 which makes ϵm positive and that in the second case ϵ is determined as for L

(x) The angles between two lines.—The axes being rectangular, let L_1 (l_1, m_1, n_1) and L_2 (l_2, m_2, n_2) be two given lines, θ_1, θ_2 the corresponding principal angles from X_1OX, ϵ_1, ϵ_2 the corresponding values of ϵ.

If $\theta_2 \geqslant \theta_1$, we call $\theta_2 - \theta_1$ the *principal angle from* L_1 *to* L_2 and denote this by $\{L_1, L_2\}$. If $\theta_2 < \theta_1$, we define $\{L_1, L_2\}$ to be $\pi + \theta_2 - \theta_1$. Thus if L_1 is parallel to L_2, $\{L_1, L_2\} = 0$; and if L_1, L_2 intersect, at R say, $\{L_1, L_2\}$ is the least angle through which L_1 must be rotated positively about R to reach the position of L_2 (Fig. 9). For non-parallel lines we have

$$\{L_1, L_2\} + \{L_2, L_1\} = \pi.$$

Any number of the form $\{L_1, L_2\} + k\pi$, k being any positive or negative integer, or zero, is called *an angle from* L_1 *to* L_2 and is denoted by $\langle L_1, L_2\rangle$, whatever the value of k. For any pair of lines we have

$$\langle L_1, L_2\rangle + \langle L_2, L_1\rangle \equiv 0 \pmod{\pi}.$$

From the formulae in part (ix) we have at once that, if $\theta_2 \geqslant \theta_1$,

$$\cos\{L_1, L_2\} = \frac{\epsilon_1\epsilon_2(l_1l_2 + m_1m_2)}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}},$$

$$\sin\{L_1, L_2\} = \frac{\epsilon_1\epsilon_2(l_1m_2 - l_2m_1)}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}},$$

and
$$\tan\{L_1, L_2\} = \frac{l_1m_2 - l_2m_1}{l_1l_2 + m_1m_2}.$$

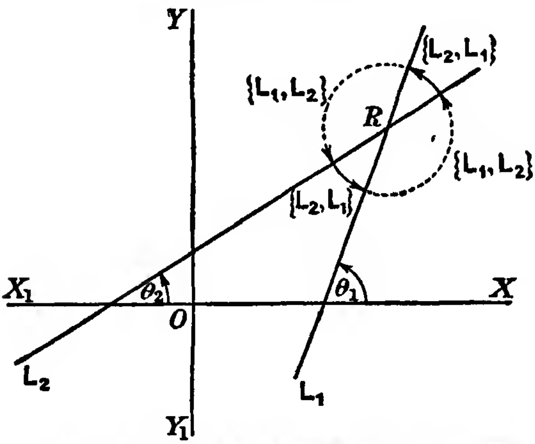


FIG 9—PRINCIPAL ANGLE BETWEEN TWO LINES.

and, if $\theta_2 < \theta_1$, the formulae are the same except that a minus sign must be put before the first two right-hand sides.

The two sets of formulae may be combined into a single set. Let $\epsilon_{12} = +1$ if $\theta_2 \geqslant \theta_1$, $\epsilon_{12} = -1$ if $\theta_2 < \theta_1$. Then

$$\cos\{L_1, L_2\} = \frac{\epsilon_{12}\epsilon_1\epsilon_2(l_1l_2 + m_1m_2)}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}},$$

$$\sin\{L_1, L_2\} = \frac{\epsilon_{12}\epsilon_1\epsilon_2(l_1m_2 - l_2m_1)}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}},$$

and
$$\tan\{L_1, L_2\} = \frac{l_1m_2 - l_2m_1}{l_1l_2 + m_1m_2}.$$

The discussion so far is rather unconventional, particularly in regard to the use of the symbols ϵ, but it is considered worth while to go into enough detail to specify a well-defined angle.

It may be proved directly or deduced immediately from the above formulae that a necessary and sufficient condition for the lines L_1, L_2 to be *perpendicular* is

$$l_1l_2 + m_1m_2 = 0.$$

Ex. 9. Prove that every line perpendicular to the line $lx + my + n = 0$ has an equation of the form $mx - ly + p = 0$, and conversely.

Ex. 10. Referring to *Ex.* 8, prove that the point on (l, m, n) nearest to (x_0, y_0) is joined to the latter point by the line $(m, -l, -mx_0 + ly_0)$ which is perpendicular to (l, m, n).

(xi) Pencils of lines.—The lines which pass through a given point O are said to form a *point-pencil* with *vertex* at the point (Fig. 10).

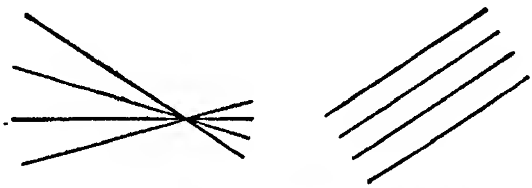


FIG. 10.—POINT-PENCIL AND PARALLEL-PENCIL.

If two of the lines have the equations

$$l_1x + m_1y + n_1 = 0,\ l_2x + m_2y + n_2 = 0,$$

then the equation

$$\lambda(l_1x + m_1y + n_1) + \mu(l_2x + m_2y + n_2) = 0$$

clearly represents a line through O. And, in fact, every line through O may have its equation represented in this way, for if $P(x_0, y_0)$ is a point other than O on such a line, the equation of OP is obtained by choosing $\lambda : \mu$ so that

$$\lambda(l_1x_0 + m_1y_0 + n_1) + \mu(l_2x_0 + m_2y_0 + n_2) = 0.$$

The lines of the point-pencil thus have co-ordinates of the form

$$(\lambda l_1 + \mu l_2,\ \lambda m_1 + \mu m_2,\ \lambda n_1 + \mu n_2),$$

and *vice versa*.

The lines parallel to a given line are said to form a *parallel-pencil* (Fig. 10). If the given line has the equation $lx + my + n$

$= 0$, then every line of the pencil has an equation of the form $lx + my + \theta = 0$, and *vice versa.* The reader will also verify easily that if two lines of the parallel-pencil are (l_1, m_1, n_1) and (l_2, m_2, n_2), then every line of the pencil has co-ordinates of the form

$$(\lambda l_1 + \mu l_2, \lambda m_1 + \mu m_2, \lambda n_1 + \mu n_2),$$

and *vice versa.*

The following interesting and useful deduction may be made.

Let the equations of the sides BC, CA, AB of a triangle be respectively $L_1 = 0$, $L_2 = 0$, $L_3 = 0$, where $L_i \equiv l_i x + m_i y + n_i$.

Then the equations

$$\mu L_2 - \nu L_3 = 0,\ \nu L_3 - \lambda L_1 = 0,\ \lambda L_1 - \mu L_2 = 0$$

represent lines through A, B, C respectively These lines are either all concurrent or all parallel, for the third equation may be expressed in the form

$$(\mu L_2 - \nu L_3) + (\nu L_3 - \lambda L_1) = 0,$$

showing that the third line belongs to the point- or parallel-pencil determined by the other two lines.

(xii) The equation of the bisector of $\{\mathsf{L}_1, \mathsf{L}_2\}$.—Let the axes be rectangular. We refer back to part (x) and consider two lines L_1, L_2 which intersect; this implies $l_1 m_2 - l_2 m_1 \neq 0$.

Every line L through the common point of L_1, L_2 has an equation of the form

$$\lambda(l_1 x + m_1 y + n_1) + \mu(l_2 x + m_2 y + n_2) = 0.$$

We determine the particular line L_0 of this system such that $\{\mathsf{L}_1, \mathsf{L}_0\} = \{\mathsf{L}_0, \mathsf{L}_1\}$ and call this the *bisector* of $\{\mathsf{L}_1, \mathsf{L}_2\}$.

Let ϵ_0, ϵ_{10}, ϵ_{02} be defined as in part (x). Then it is easy to see that if $\theta_2 > \theta_1$, $\epsilon_{10} = \epsilon_{02}$, and if $\theta_2 < \theta_1$, $\epsilon_{10} = -\epsilon_{02}$.

We have

$$\sin\{\mathsf{L}_1, \mathsf{L}_0\} = \sin\{\mathsf{L}_0, \mathsf{L}_2\},$$

giving

$$\frac{\epsilon_{10}\epsilon_1\epsilon_0\{l_1(\lambda m_1 + \mu m_2) - (\lambda l_1 + \mu l_2)m_1\}}{\sqrt{(l_1^2 + m_1^2)\{(\lambda l_1 + \mu l_2)^2 + (\lambda m_1 + \mu m_2)^2\}}} = \frac{\epsilon_{02}\epsilon_0\epsilon_2\{(\lambda l_1 + \mu l_2)m_2 - l_2(\lambda m_1 + \mu m_2)\}}{\sqrt{(l_2^2 + m_2^2)\{(\lambda l_1 + \mu l_2)^2 + (\lambda m_1 + \mu m_2)^2\}}}.$$

Hence, after a little straight-forward reduction,

$$\lambda : \mu = \frac{\epsilon_{10}\epsilon_1}{\sqrt{l_1^2 + m_1^2}} : \frac{\epsilon_{02}\epsilon_2}{\sqrt{l_2^2 + m_2^2}}.$$

The equation of the bisector of $\{\mathsf{L}_1, \mathsf{L}_2\}$ is therefore

$$\frac{\epsilon_1(l_1x + m_1y + n_1)}{\sqrt{l_1^2 + m_1^2}} \pm \frac{\epsilon_2(l_2x + m_2y + n_2)}{\sqrt{l_2^2 + m_2^2}} = 0,$$

where the positive sign is taken if $\theta_2 > \theta_1$, and the negative sign is taken if $\theta_2 < \theta_1$.

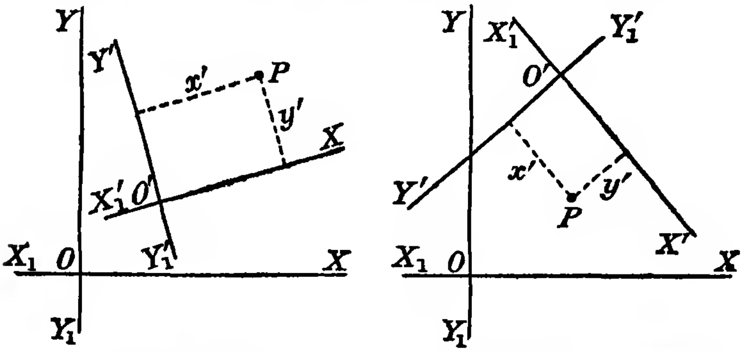
Whichever equation is taken according to this rule, it is easy to see that the other equation represents the bisector of $\{\mathsf{L}_2, \mathsf{L}_1\}$. The two bisectors are perpendicular.

Ex. 11. A, B, C are the points (2, 2), (0, 0), (5, 0) respectively. Take L_1, L_2, L_3 to be BC, CA, AB and verify that the interior angles at A, B, C are respectively $\{\mathsf{L}_3, -\mathsf{L}_2\}$, $\{\mathsf{L}_1, \mathsf{L}_3\}$, $\{\mathsf{L}_2, -\mathsf{L}_1\}$. Prove that the bisectors of these angles are concurrent, using the result at the end of part (xi) of this section.

Prove also that the bisectors of $\{\mathsf{L}_1, \mathsf{L}_2\}$, $\{\mathsf{L}_3, \mathsf{L}_1\}$, $\{\mathsf{L}_3, \mathsf{L}_2\}$ are concurrent; and obtain two similar results.

(xiii) Change of axes.

(*a*) Let X_1OX, Y_1OY and $X_1'O'X'$, $Y_1'O'Y'$ be two pairs of rectangular axes and let a point P have distance-co-ordinates (x, y) relative to the first pair and (x', y') relative to the second pair. We seek formulae expressing x', y' in terms of x, y.



Same sense *Opposite senses*

Fig. 11.—Change of Rectangular Axes.

Positive rotations of the plane relative to the two pairs of axes may have the same sense or opposite senses; in the first case we say that the two pairs of axes have the *same sense*, and in the second case that they have *opposite senses* (Fig. 11).

Let the equations of $X_1'O'X'$, $Y_1'O'Y'$ relative to the first pair of axes be respectively $lx + my + n = 0$, $l'x + m'y + n' = 0$, where, of course, $ll' + mm' = 0$. We choose ϵ, ϵ' each to be $+ 1$

or -1 in such a manner that Y' is on the positive side of $(\epsilon l, \epsilon m, \epsilon n)$ and X' is on the positive side of $(\epsilon' l', \epsilon' m', \epsilon' n')$. Then x' is the algebraic distance of P from $(\epsilon' l', \epsilon' m', \epsilon' n')$; therefore

$$x' = \epsilon'(l'x + m'y + n')/\sqrt{l'^2 + m'^2}.$$

Similarly,

$$y' = \epsilon(lx + my + n)/\sqrt{l^2 + m^2}.$$

These formulae are called *formulae for change of axes.* They may evidently be put in the form

$$x' = \lambda x + \mu y + \nu,$$
$$y' = \lambda' x + \mu' y + \nu',$$

with

$$\lambda^2 + \mu^2 = 1, \ \lambda'^2 + \mu'^2 = 1, \ \lambda\lambda' + \mu\mu' = 0;$$

(*b*) Next, let X_1OX, Y_1OY be a pair of oblique axes and let X_1OX, H_1OH be a pair of rectangular axes, with H, Y on the same side of X_1OX (Fig. 12). Relative to either pair of axes, let a positive rotation through an angle ω carry OX into OY. Then, if a point has distance-co-ordinates (x, y) relative to the first pair of axes and (ξ, η) relative to the second pair, we have at once the relations:

$$\xi = x + y \cos \omega,$$
$$\eta = \quad y \sin \omega,$$

which are equivalent to

$$x = \xi - \eta \cot \omega,$$
$$y = \quad \eta \operatorname{cosec} \omega.$$

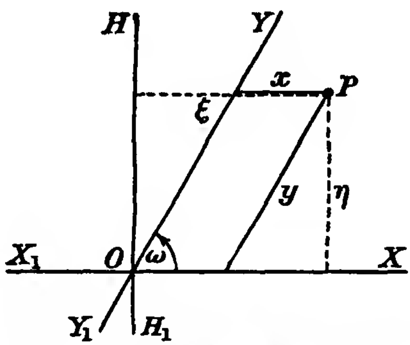


FIG. 12—CHANGE FROM RECTANGULAR TO OBLIQUE AXES.

(*c*) We now combine the results of (*a*) and (*b*).

Let X_1OX, Y_1OY and $X_1'O'X'$, $Y_1'O'Y'$ be two pairs of axes, with the same or opposite senses, and each oblique or rectangular. Let any point P have distance-co-ordinates (x, y) relative to the first pair and (x', y') relative to the second pair.

Let X_1OX, H_1OH be a pair of rectangular axes associated with X_1OX, Y_1OY as in (*b*); and let the distance-co-ordinates of P relative to these new axes be (ξ, η).

Then, if the lines $X_1'O'X'$, $Y_1'O'Y'$ have the equations

$$lx + my + n = 0,$$
$$l'x + m'y + n' = 0$$

respectively relative to X_1OX, Y_1OY, they have the equations

$$l(\xi - \eta \cot \omega) + m(\eta \operatorname{cosec} \omega) + n = 0,$$
$$l'(\xi - \eta \cot \omega) + m'(\eta \operatorname{cosec} \omega) + n' = 0$$

respectively relative to X_1OX, H_1OH.

Now the perpendicular distances of P from $X_1'O'X'$, $Y_1'O'Y'$ are respectively $|y' \sin \omega'|$ and $|x' \sin \omega'|$, where a positive rotation through an angle ω' relative to the axes $X_1'O'X'$, $Y_1'O'Y'$ carries $O'X'$ into $O'Y'$; hence

$$x' \sin \omega' = \frac{\epsilon'\{l'(\xi - \eta \cot \omega) + m'(\eta \operatorname{cosec} \omega) + n'\}}{\sqrt{l'^2 + (-l' \cot \omega + m' \operatorname{cosec} \omega)^2}},$$

giving

$$x' = \frac{\epsilon'(l'x + m'y + n')}{\sqrt{l'^2 - 2l'm' \cos \omega + m'^2}} \cdot \frac{\sin \omega}{\sin \omega'};$$

similarly

$$y' = \frac{\epsilon(lx + my + n)}{\sqrt{l^2 - 2lm \cos \omega + m^2}} \cdot \frac{\sin \omega}{\sin \omega'};$$

and in these formulae ϵ, ϵ' are given each the value $+1$ or -1, which makes the co-ordinates of X', Y' positive.

The importance of these formulae in regard to further developments is that x', y' are expressed *linearly* in terms of x, y. This fact alone, if granted *a priori* or proved otherwise, shows at once that, since x' is zero along $Y_1'O'Y'$, x' is proportional to $l'x + m'y + n'$; and similarly for y'.

(xiv) Change of distance-scale.—The axes being oblique or rectangular, let the distance co-ordinates of a point P be x, y.

The equations $x' = ax$, $y' = by$, where a, b are non-zero real numbers, determine a new (1, 1) correspondence between the points of the plane and the ordered pairs of real numbers. We refer to x', y' as a pair of *algebraic co-ordinates* of P.

In speaking of distance co-ordinates we have hitherto assumed the existence of one scale for measuring all distances. The introduction of a system of algebraic co-ordinates is equivalent to the choice of new units of distance for measurements of sensed distances parallel to X_1OX and Y_1OY. Thus, if A is the point given by $x = 1$, $y = 0$, and A' is the point given by $x' = 1$, $y' = 0$, we have $\overrightarrow{OA'}_{OX} = a\overrightarrow{OA}_{OX}$.

With a system of algebraic co-ordinates, the equation of a line is still linear in the co-ordinates, and *vice versa*. In fact, with the present notation, the line represented by $lx + my + n = 0$ is also represented by $l'x' + m'y' + n' = 0$, where $l' : m' : n' = l/a : m/b : n$. We call l', m', n' a set of algebraic co-ordinates of the line and refer to the line as (l', m', n') in the new system.

(xv) Change of algebraic co-ordinate systems.—Let X_1OX, Y_1OY and $X_1'O'X'$, $Y_1'O'Y'$ be two pairs of axes, oblique or rectangular; with certain scales of measurement let a point P have algebraic co-ordinates (x, y) relative to the first pair and (x', y') relative to the second pair. We express x', y' in terms of x, y.

Let the equations of $X_1'O'X'$, $Y_1'O'Y'$ relative to the first pair of axes and scales of measurement be respectively

$$lx + my + n = 0, \; l'x + m'y + n' = 0.$$

We choose a standard scale of measurement for all distances, with this and the first pair of axes let P have co-ordinates (ξ, η); and with this and the second pair let P have co-ordinates (ξ', η'). Then

$$\begin{aligned} x &= a\xi, \quad & y &= b\eta, \\ x' &= a'\xi', \quad & y' &= b'\eta' \end{aligned}$$

for some a, b, a', b' The equations of $X_1'O'X'$, $Y_1'O'Y'$ relative to the first pair of axes and the new scale become respectively

$$la\xi + mb\eta + n = 0, \quad l'a\xi + m'b\eta + n' = 0.$$

By part (xiii) (c) of this section

$$\begin{aligned} \xi' &= k'(l'a\xi + m'b\eta + n'), \\ \eta' &= k(la\xi + mb\eta + n), \end{aligned}$$

for some k, k'.

Hence

$$\begin{aligned} x' &= a'k'(l'x + m'y + n'), \\ y' &= b'k(lx + my + n) \end{aligned}$$

Thus x', y' are expressed *linearly* in terms of x, y.

Conversely, every transformation of the form

$$\begin{aligned} x' &= \lambda'x + \mu'y + \nu', \\ y' &= \lambda x + \mu y + \nu, \end{aligned}$$

with real numbers, where $\lambda\mu' \neq \lambda'\mu$, $\lambda^2 + \mu^2 \neq 0$, $\lambda'^2 + \mu'^2 \neq 0$, represents a change of algebraic co-ordinate systems

Let x, y be algebraic co-ordinates relative to axes X_1OX, Y_1OY and certain scales of measurement. Then the equations $\lambda x + \mu y + \nu = 0$, $\lambda'x + \mu'y + \nu' = 0$, with the conditions imposed, represent two intersecting lines, on these we take points X_1', O', X' belonging to the first and Y_1', O', Y' belonging to the second, the common point O' being between X_1', X' and between Y_1', Y'.

With $X_1'O'X'$ and $Y_1'O'Y'$ as axes and arbitrary scales of measurement let ξ', η' be the co-ordinates of the point represented by (x, y) relative to X_1OX, Y_1OY. Then, as we have seen above,

$$\begin{aligned} \xi' &= h'(\lambda'x + \mu'y + \nu'), \\ \eta' &= h(\lambda x + \mu y + \nu), \end{aligned}$$

for some h, h'.

Hence $x' = \xi'/h'$, $y' = \eta'/h$; and therefore x', y' are algebraic co-ordinates relative to $X_1'O'X'$, $Y_1'O'Y'$.

(xvi) Distance as an invariant. (*a*) If two points P_1, P_2 have respectively the distance co-ordinates (x_1, y_1), (x_2, y_2) relative to one pair of rectangular axes, and (x_1', y_1'), (x_2', y_2') relative to another pair, then

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2,$$

both sides of this equality being equal to $|P_1P_2|^2$.

The two sets of co-ordinates are related by formulae of the form (see part (xiii) (*a*) of this section)

$$x' = \lambda x + \mu y + \nu,\ y' = \lambda' x + \mu' y + \nu',$$

where $\lambda^2 + \mu^2 = \lambda'^2 + \mu'^2 = 1$, $\lambda\lambda' + \mu\mu' = 0$.

From an algebraic point of view, we say that $|P_1P_2|$, and equally the function $(x_1 - x_2)^2 + (y_1 - y_2)^2$, is an *invariant* with respect to the transformation expressed by the above equations.

(*b*) We prove now the converse theorem that if the function $(x_1 - x_2)^2 + (y_1 - y_2)^2$ is invariant with respect to the transformation

$$x' = \alpha x + \beta y + \gamma,\quad y' = \alpha' x + \beta' y + \gamma',$$

then, if (x, y) are taken to represent the distance co-ordinates of any point relative to one pair of rectangular axes, (x', y') represent the distance co-ordinates of the same point relative to another pair of rectangular axes.

We have

$$\begin{aligned}
&(x_1' - x_2')^2 + (y_1' - y_2')^2 \\
&= \{\alpha(x_1 - x_2) + \beta(y_1 - y_2)\}^2 + \{\alpha'(x_1 - x_2)^2 + \beta'(y_1 - y_2)^2\} \\
&= (\alpha^2 + \alpha'^2)(x_1 - x_2)^2 + (\beta^2 + \beta'^2)(y_1 - y_2)^2 \\
&\qquad + 2(\alpha\beta + \alpha'\beta')(x_1 - x_2)(y_1 - y_2).
\end{aligned}$$

Therefore, by the invariance property,

$$\begin{aligned}
\alpha^2 + \alpha'^2 &= 1 \\
\beta^2 + \beta'^2 &= 1 \\
\alpha\beta + \alpha'\beta' &= 0.
\end{aligned}$$

Hence,

$$\alpha^2 = \alpha^2(\beta^2 + \beta'^2) = \alpha^2\beta^2 + \alpha^2\beta'^2 = \alpha'^2\beta'^2 + \alpha^2\beta'^2 = \beta'^2(\alpha^2 + \alpha'^2) = \beta'^2,$$

and, similarly, $\beta^2 = \alpha'^2$. Therefore

$$\begin{aligned}
\alpha^2 + \beta^2 &= 1, \\
\alpha'^2 + \beta'^2 &= 1.
\end{aligned}$$

Further,

$$(\alpha\beta' - \alpha'\beta)^2 = (\alpha^2 + \alpha'^2)(\beta^2 + \beta'^2) - (\alpha\beta + \alpha'\beta')^2$$
$$= 1,$$

so that

$$\alpha\beta' - \alpha'\beta = \pm 1.$$

Also,

$$(\alpha\alpha' + \beta\beta')^2 = (\alpha^2 + \beta^2)(\alpha'^2 + \beta'^2) - (\alpha\beta' - \alpha'\beta)^2$$
$$= 0.$$

Therefore

$$\alpha\alpha' + \beta\beta' = 0.$$

The coefficients in the equations of transformation thus satisfy the conditions that the equations represent a change of distance co-ordinates from one pair of rectangular axes to another.

(c) Some simplification of all the relations appears by the use of trigonometric functions.

Since $\alpha^2 + \beta^2 = 1$, we may define a number θ, $0 \leqslant \theta < 2\pi$, by the equations $\cos\theta = \alpha$, $\sin\theta = \beta$; and since $\alpha'^2 + \beta'^2 = 1$, we may define a number ϕ, $0 \leqslant \phi < 2\pi$, by the equations $\cos\phi = \alpha'$, $\cos\phi = \beta'$.

Then $\cos(\phi - \theta) = \alpha\alpha' + \beta\beta' = 0$, $\sin(\phi - \theta) = \alpha\beta' - \alpha'\beta = \pm 1$. Taking the positive sign, $\phi - \theta = \pi/2$, and the equations of transformation become

$$x' = x\cos\theta + y\sin\theta + \gamma,$$
$$y' = -x\sin\theta + y\cos\theta + \gamma'.$$

Taking the negative sign, $\phi - \theta = -\pi/2$, and the equations of transformation become

$$x' = x\cos\theta + y\sin\theta + \gamma,$$
$$y' = x\sin\theta - y\cos\theta + \gamma'.$$

A transformation of the first kind is called *positive*; one of the second kind is called *negative*. The distinction between the two kinds of transformation lies geometrically in the fact that in the positive case the two pairs of axes have the same sense, whereas in the negative case the two pairs of axes have opposite senses; the proof is straightforward.

2. The complex euclidean plane E_c.

(i) Explanatory remarks.—The preliminary aspects of geometry so far encountered have involved linear equations. However, much of geometry involves the study of non-linear equations and in connection with these we meet a difficulty which may be illustrated by the following example.

In seeking the points, if any, common to the circle $x^2 + y^2 = 1$ and the line $x = k$, we are led to the equation in y, obtained by eliminating x from these equations, $y^2 = 1 - k^2$. Remembering that we are restricted to real numbers, we see that there are two common points if $|k| \leqslant 1$, these coinciding in the case $|k| = 1$, and none if $|k| > 1$ (Fig 13).

The algebraic theorem involved is that an equation of degree n, with real coefficients, has n *or fewer* real roots. The continued application of this theorem leads to many complications consequent on the words "or fewer." It is therefore considered worth while to take advantage of a form of the so-called Funda-

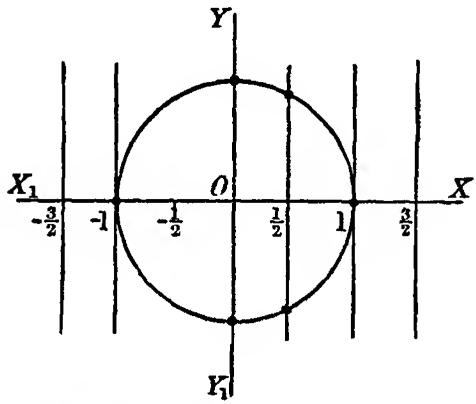


FIG 13.—VARIABLE NUMBER OF INTERSECTIONS OF A REAL LINE WITH A REAL CIRCLE

mental Theorem of Algebra that every equation of degree n, with real or complex coefficients, has precisely n real or complex roots, these being properly counted

In order to have this advantage, we introduce new entities, called *unreal*, into geometry, the existing entities are called *real*. It then becomes possible, for example, to say that every line meets a circle in two points, real or unreal.

(ii) Unreal points.—Let E_R be a euclidean plane. With any system of co-ordinates, S say, every (real) point in E_R can be represented by an ordered pair of real numbers x, y, its co-ordinates in that system.

Every ordered pair of real or complex numbers, at least one of which is not real, is called an *unreal element attached to* E_R *through* S. The two numbers are called the *co-ordinates* of the unreal element relative to S.

Consider now any two co-ordinate systems S, S′. The (real)

co-ordinates (x, y) and (x', y') of a point relative to S, S′ respectively are connected by equations of the form

$$x' = \lambda x + \mu y + \nu, \quad y' = \lambda' x + \mu' y + \nu',$$

where $\lambda, \mu, \nu, \lambda', \mu', \nu'$ are real numbers such that $\lambda^2 + \mu^2 \neq 0$, $\lambda'^2 + \mu'^2 \neq 0$, $\lambda\mu' \neq \lambda'\mu$.

The unreal elements (x_1, y_1) attached to E_R through S and (x_2, y_2) attached to E_R through S′ are called *equivalent* when

$$x_2 = \lambda x_1 + \mu y_1 + \nu, \quad y_2 = \lambda' x_1 + \mu' y_1 + \nu'.$$

It is simple to prove that every unreal element attached to E_R through S is equivalent to one and only one unreal element attached to E_R through S′, and *vice versa*; the proof is left to the reader.

The aggregate of all unreal elements attached to E_R through all co-ordinate systems and which are equivalent to a given unreal element is called an *unreal point attached to E_R*. If (x, y) is the unreal element thus associated in S with an unreal point P, we call x, y the *co-ordinates of* P *relative to* S.

The points of E_R are called *real points* to distinguish them from the unreal points.

The set of (real) points of E_R plus the set of unreal points attached to E_R is called the *complex euclidean plane E_C covering E_R*; E_R is said to be *embedded* in E_C. E_R is called a *real euclidean plane* in contrast to E_C.

(iii) Lines in the complex euclidean plane.—The set of real and unreal points in E_C whose co-ordinates relative to a given co-ordinate system S satisfy an equation of the form

$$lx + my + n = 0,$$

where l, m, n are real or complex and l, m are not both zero, is called a *line*. The linear equation is called here *affine* in virtue of the restriction on l, m.

The reader will observe that the word "line" is here used in a new sense and has a different meaning from that which it stands for in the case of a real euclidean plane. And it is worth noting that, with complex numbers, the inequality $l^2 + m^2 \neq 0$ is *not* equivalent to saying that l, m are not both zero.

The definition of a line, just given, is expressed with reference to a particular co-ordinate system S. It is most significant that the definition is unaltered if we replace S by any other co-ordinate system S′; and this we now prove.

The co-ordinates (x, y) of a point, real or unreal, relative to S are related to its co-ordinates (x', y') relative to S′ by equations of the form

$$x = \alpha x' + \beta y' + \gamma, \quad y = \alpha' x' + \beta' y' + \gamma',$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are real and α, β not both zero, α', β' not both zero, and $\alpha\beta' - \alpha'\beta \neq 0$.

Hence, the co-ordinates relative to S′ of the points on the line $lx + my + n = 0$, are those and only those which satisfy the equation

$$(\alpha l + \alpha' m)x' + (\beta l + \beta' m)y' + \gamma l + \gamma' m + n = 0,$$

which is linear and affine, since if $\alpha l + \alpha' m = \beta l + \beta' m = 0$ we should have $\alpha\beta' - \alpha'\beta = 0$.

Moreover, the points whose co-ordinates relative to S′ satisfy the affine equation

$$l'x' + m'y' + n' = 0$$

are those and only those whose co-ordinates relative to S satisfy the equation

$$(\beta' l' - \alpha' m')x + (\alpha m' - \beta l')y + (\beta\gamma' - \beta'\gamma)l + (\alpha'\gamma - \alpha\gamma')m + (\alpha\beta' - \alpha'\beta)n = 0,$$

which is also affine; this last equation is obtained by reversing the equations for change of co-ordinates.

Thus, just as in the case of a real euclidean plane, an affine linear equation represents a line in any co-ordinate system, and *vice versa.*

The numbers l, m, n are called a set of *co-ordinates* of the line relative to the co-ordinate system S, and we refer to the line by the ordered triad (l, m, n). Defining a *complex terset* in a similar manner to a real terset, we observe that the lines of E_C are in (1, 1) correspondence with all complex affine tersets.

(iv) Joachimstal's formulae.—Formulae analogous to those of Joachimstal for a real euclidean plane are obtained below for a complex euclidean plane. It should be remarked that the notion of distance is not used. The same reasoning may be employed in regard to a real euclidean plane but the interpretation of the results will then be less specific than in the Joachimstal formulae already given.

Let (x_1, y_1) and (x_2, y_2) be any two distinct points in E_C. There is just one line containing these; its equation is

$$lx + my + n = 0,$$

where

$$lx_1 + my_1 + n = 0,$$
$$lx_2 + my_2 + n = 0,$$

and therefore

$$l : m : n = y_1 - y_2 : x_2 - x_1 : x_1y_2 - x_2y_1.$$

The equation of the line may therefore be put in either of the equivalent forms

$$x(y_1 - y_2) + y(x_2 - x_1) + (x_1y_2 - x_2y_1) = 0,$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

We have, further, for all (real or complex) values of λ, μ,

$$l(\lambda x_1 + \mu x_2) + m(\lambda y_1 + \mu y_2) + n(\lambda + \mu) = 0.$$

Now l, m are not both zero; suppose $l \neq 0$ (the argument is similar if $m \neq 0$) Then, for any given point (x, y) on the line, we may choose $\lambda : \mu$ so that

$$y = (\lambda y_1 + \mu y_2)/(\lambda + \mu);$$

this evidently involves $\lambda + \mu \neq 0$. With such a ratio for $\lambda : \mu$, we have

$$lx(\lambda + \mu) = l(\lambda x_1 + \mu x_2),$$

and therefore, since $l \neq 0$,

$$x = (\lambda x_1 + \mu x_2)/(\lambda + \mu).$$

Thus, the co-ordinates (x, y) of every point on the line may be expressed in the form

$$x = \frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \quad y = \frac{\lambda y_1 + \mu y_2}{\lambda + \mu},$$

with $\lambda + \mu \neq 0$. These are analogous to Joachimstal's formulae

It is left to the reader to verify that one and only one point on the line corresponds to a given value of the ratio $\lambda : \mu$ (excluding $1 : -1$ and $0 : 0$), and *vice versa*, that the point (x_1, y_1) arises when $\lambda : \mu = 1 : 0$; and that the point (x_2, y_2) arises when $\lambda : \mu = 0 : 1$

Ex 1. A necessary and sufficient condition for the points (x_1, y_1), (x_2, y_2), (x_3, y_3) to be in line is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

(v) Parallel lines. Pencils of lines.—In any co-ordinate system for E_C, the equations

$$l_1x + m_1y + n_1 = 0, \quad l_2x + m_2y + n_2 = 0$$

represent different lines if and only if

$$l_1 : m_1 : n_1 \neq l_2 : m_2 : n_2.$$

When this inequality is satisfied, the two lines meet in the point given by

$$x : y : 1 = m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1$$

except and only except when $l_1m_2 - l_2m_1 = 0$. If $l_1m_2 = l_2m_1$ the two lines have no point in common and are said to be *parallel*.

Point-pencils and *parallel-pencils* of lines are defined as for a real euclidean plane. It may be proved as there that the lines of a point- or parallel-pencil determined by the lines (l_1, m_1, n_1), (l_2, m_2, n_2) have co-ordinates of the form

$$(\lambda l_1 + \mu l_2, \quad \lambda m_1 + \mu m_2, \quad \lambda n_1 + \mu n_2)$$

and *vice versa*.

(vi) Conjugate points and lines.—We follow the usual notation, using a, $\bar{a}$ to denote conjugate complex numbers.

It is easy to verify that if two points have co-ordinates of the form (x, y), $(\bar{x}, \bar{y})$ in one system of co-ordinates, then they have co-ordinates of the same form in any other system of co-ordinates. Two such points are called *conjugate*. A point of E_C is self-conjugate if and only if it belongs to E_R.

Similarly, if two lines have co-ordinates of the form (l, m, n), $(\bar{l}, \bar{m}, \bar{n})$ in one system of co-ordinates, then they have co-ordinates of the same form in any other system of co-ordinates. Two such lines are called *conjugate*.

A line (l, m, n) is said to be *real* if l, m, n are proportional to real numbers; otherwise the line is called *unreal*. A real line of E_C must however not be confused with a line of E_R; a real line of E_C contains all the points of a line of E_R but contains besides a set of pairs of conjugate unreal points of E_C. For example, the real line $x = 0$ of E_C contains all the real points which constitute the y-axis of E_R and also all unreal points of the form $(0, y' + iy'')$, where y', y'' are any real numbers ($y'' \neq 0$). A real line is self-conjugate.

Two real points of E_C are joined by a real line; so are two conjugate unreal points. Two non-parallel real lines of E_C meet in a real point; so do two non-parallel conjugate unreal lines. The proofs are trivial.

Equally simple to prove are the following statements. If an unreal point lies on a real line so does its conjugate unreal point. If an unreal line contains a real point so does its conjugate unreal line. The real lines of E_C are in (1, 1) correspondence with, and cover, the lines of E_R.

(vii) Perpendicular lines.—In any co-ordinate system, the lines $x = 0$, $y = 0$ of E_C contain the axes of the subsidiary co-ordinate

system in E_R. These lines are called the *axes* of the co-ordinate system in E_C and are said to be *rectangular* when the corresponding axes in E_R are rectangular; the co-ordinate system in E_C is then also called *rectangular*.

Let us now suppose that two lines have co-ordinates (l_1, m_1, n_1), (l_2, m_2, n_2) in a rectangular co-ordinate system in E_C. If $l_1l_2 + m_1m_2 = 0$ the lines are said to be *perpendicular*. Then it is clear that if these lines are real, the lines of E_R which they contain are also perpendicular and *vice versa*. For the definition to be significant we need to show that, if two lines are perpendicular relative to one rectangular co-ordinate system S in E_C, they are also perpendicular relative to any other rectangular co-ordinate system S′.

The equations for transforming point-co-ordinates from those in S to those in S′ are (by section 1, part (xiii)) of the form

$$x = \lambda x' + \mu y' + \nu,\ y = \lambda' x' + \mu' y' + \nu',$$

where $$\lambda^2 + \mu^2 = \lambda'^2 + \mu'^2 = 1,\ \lambda\lambda' + \mu\mu' = 0.$$

If then the lines have co-ordinates (l_1, m_1, n_1), (l_2, m_2, n_2) relative to S, they have co-ordinates (l_1', m_1', n_1'), (l_2', m_2', n_2') relative to S′, where

$$\begin{aligned} l_1' &= l_1\lambda + m_1\lambda', & m_1' &= l_1\mu + m_1\mu', & n_1' &= \nu l_1 + \nu' m_1 + n_1, \\ l_2' &= l_2\lambda + m_2\lambda', & m_2' &= l_2\mu + m_2\mu', & n_2' &= \nu l_2 + \nu' m_2 + n_2. \end{aligned}$$

Therefore

$$\begin{aligned} & l_1'l_2' + m_1'm_2' \\ &= (l_1\lambda + m_1\lambda')(l_2\lambda + m_2\lambda') + (l_1\mu + m_1\mu')(l_2\mu + m_2\mu') \\ &= l_1l_2(\lambda^2 + \mu^2) + (l_1m_2 + l_2m_1)(\lambda\lambda' + \mu\mu') + m_1m_2(\lambda'^2 + \mu'^2) \\ &= l_1l_2 + m_1m_2, \end{aligned}$$

and from this our theorem follows.

It is worth remarking that in E_C, but not in E_R, a line may be perpendicular to itself. Every such line is easily shown to have an equation of the form

$$y - y_0 = i(x - x_0), \text{ or } y - y_0 = -i(x - x_0);$$

the two lines whose equations are written are called the *isotropic lines* through the point (x_0, y_0). All the isotropic lines form two parallel-pencils: those for which $l : m = i : 1$ and those for which $l : m = -i : 1$.

(viii) Distance-function.—The general concept of distance (unsensed) between two points in an abstract space is as follows.

A real number $d(P, Q)$ is defined, if possible, for every pair of points P, Q in such a way that

(a) $d(P, Q) = d(Q, P)$, so that the number is independent of the order in which the points are named;

(b) $d(P, Q) = 0$ if and only if P, Q denote the same point;

(c) $d(P, Q) + d(Q, R) \geqslant d(P, R)$ for any three points P, Q, R.

It follows then that

$$d(P, Q) = \tfrac{1}{2}\{d(P, Q) + d(Q, P)\} \geqslant \tfrac{1}{2}d(P, P);$$

and therefore, since $d(P, P) = 0$, $d(P, Q) \geqslant 0$.

The number $d(P, Q)$ is called a *distance-function.*

In the case of a real euclidean plane E_R such a distance-function is determined by taking

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where P, Q have distance co-ordinates (x_1, y_1), (x_2, y_2) respectively in any rectangular system.

It is natural to seek a distance-function for the complex euclidean plane E_C. The formula just given for E_R does not determine a distance-function for E_C since, for example, the second requirement for a distance-function is not fulfilled, as may be seen by taking P, Q to be respectively the points $(1, 0)$, $(0, i)$.

A distance-function for E_C may, however, be determined relative to any particular co-ordinate system S as follows. Let the co-ordinates of P, Q be respectively $(x_1' + ix_1'', y_1' + iy_1'')$, $(x_2' + ix_2'', y_2' + iy_2'')$, where $x_1', x_1'', \ldots, y_2''$ are real numbers. We then take

$$d(P, Q) = \sqrt{(x_1' - x_2')^2 + (x_1'' - x_2'')^2 + (y_1' - y_2')^2 + (y_1'' - y_2'')^2}.$$

The first two requirements are obviously satisfied by $d(P, Q)$ That the third is satisfied follows from the inequality for real numbers

$$\sqrt{a^2 + b^2 + c^2 + d^2} + \sqrt{A^2 + B^2 + C^2 + D^2}$$
$$\geqslant \sqrt{(a + A)^2 + (b + B)^2 + (c + C)^2 + (d + D)^2}$$

by putting $a = x_1' - x_2'$, $A = x_2' - x_3'$, *etc*

It may be accepted that the notion of distance-function is both more complicated and less useful in the case of a complex euclidean plane than in the case of a real euclidean plane. We do not develop the idea here, nor do we need it in what follows.

3. The modified euclidean plane (real or complex) E_M.

(i) Explanatory remarks.—In section 2 we introduced the concept of unreal points in order to facilitate the geometrical interpretation of algebraic equations. There is still, however, a limitation on the interpretation of such equations which applies equally to the real and to the complex euclidean plane. Geometrically this limitation corresponds to the possibility of lines being parallel instead of intersecting, algebraically it corresponds to the fact that every point has finite co-ordinates.

In order to avoid this limitation, we find it convenient to amplify in a new way the basic set of geometrical entities thereby creating a new kind of plane in which it may be said, with due regard to the extended meaning of certain words, that every line intersects every other line.

In this section we use the letter E to denote a real or complex euclidean plane, if we regard E as real, the corresponding algebra refers only to real numbers; and if we regard E as complex, the algebra refers to real and complex numbers.

(ii) Inaccessible points.—The aggregate of all lines parallel to and including a given line of E is called an *inaccessible point*. The points of E are, for distinction, called *accessible*.

An inaccessible point is determined equally well by any line of the aggregate. It is said to *lie on* every such line; and every such line is said to *pass through* the inaccessible point.

The aggregate of accessible and inaccessible points is called a *modified euclidean plane* and denoted by E_M. E is said to be *embedded* in E_M, and E_M to *cover* E.

Every line of E together with the inaccessible point which it determines is called a *line* of E_M. It follows that any two lines in E_M have just one point, accessible or inaccessible, in common; and that any two points of E_M lie on just one line, unless both points are inaccessible. To remove this exception, we call the set of inaccessible points a line also and name this the *inaccessible line* of E_M

(iii) Modified co-ordinates.—Our object is now to set up a system of co-ordinates in E_M which apply simultaneously both to accessible and inaccessible points.

Let S be any system of co-ordinates in E; relative to S let the co-ordinates of any point P of E be (x, y). Then any ordered triad of numbers (X, Y, Z), such that

$$Z \neq 0, \quad x = X/Z, \quad y = Y/Z,$$

is called a set of *modified co-ordinates* of P relative to S. Given P, these triads are all determined and differ only in respect of a

factor of proportionality; and, given any such triad, one accessible point is determined, namely the point $(X/Z,\ Y/Z)$ relative to S.

We must obviously arrange that the triads (X, Y, Z) for which $Z = 0$ should be attached to the inaccessible points. We shall find that the triad $(0, 0, 0)$ is left over: there is no point, accessible or inaccessible, with modified co-ordinates $(0, 0, 0)$.

To determine an inaccessible point it is enough to specify the line through the origin which belongs to its aggregate of parallels; and for this it is enough to give the co-ordinates x, y of any point on this line other than the origin, thus x and y are not both simultaneously zero. If we multiply x and y by a common factor, the point (x, y) varies on the selected line through the origin Thus only the ratio $x : y$ depends on the inaccessible point and *vice versa*. We therefore define the triad $(x, y, 0)$ to be a set of *modified co-ordinates for the inaccessible point* relative to S.

We have thus an algebraic means of determining any point of E_M irrespective of whether it is accessible or inaccessible, namely an ordered triad of numbers (X, Y, Z), which are always finite and not all zero, and which can be varied by a common multiplier without the point being changed. The accessible points are distinguished from the inaccessible points only by the fact that for the former $Z \neq 0$, while for the latter $Z = 0$.

The points of E_M are, in fact, in $(1, 1)$ correspondence with all the tersets $[X, Y, Z]$, excluding $[0, 0, 0]$.

(iv) The equation of a line in E_M.

(*a*) The co-ordinates (x, y) of every accessible point on a line in E_M, excluding the inaccessible line, satisfy an equation of the form

$$lx + my + n = 0,$$

with l, m, not both zero; and *vice versa*. The modified co-ordinates of these points therefore satisfy

$$lX + mY + nZ = 0,\ Z \neq 0.$$

A triad of modified co-ordinates for the inaccessible point on the line is easily seen to be $(m, -l, 0)$; and these satisfy

$$lX + mY + nZ = 0,\ Z = 0.$$

The equation

$$lX + mY + nZ = 0$$

thus represents the whole line in E_M.

When l, m are both zero, but n is not zero, the equation $lX + mY + nZ = 0$ becomes $Z = 0$, which is the equation

satisfied by the co-ordinates of the inaccessible points and these only.

Hence, every line in E_M is represented by an equation of the form

$$lX + mY + nZ = 0,$$

with l, m, n not all zero; and *vice versa.*

We call the triad (l, m, n) a set of *co-ordinates of the line* relative to the system of modified co-ordinates for points. Evidently, then, in E_M the lines are in (1, 1) correspondence with all the tersets $[l, m, n]$, excluding $[0, 0, 0]$.

(*b*) If, in the equation $lX + mY + nZ = 0$, we regard l, m, n and X, Y, Z both as given triads, we have the *condition of incidence* of a certain line and a certain point.

If we regard only l, m, n as fixed, we have the equation satisfied by the co-ordinates of every point on a certain line; we call it the *point-equation of the line* (l, m, n).

If we regard only X, Y, Z as fixed, we have the equation satisfied by the co-ordinates of every line through a certain point; we call it the *line-equation of the point* (X, Y, Z).

Because (kX, kY, kZ), $k \neq 0$, are co-ordinates of (X, Y, Z), independently of the value of k, we speak of (X, Y, Z) as a set of *homogeneous co-ordinates* of the point. Similarly, we speak of (l, m, n) as a set of homogeneous co-ordinates.

(*c*) The following results for E_M may be proved much as are the corresponding results for E.

Ex. 1. The point-equation of the line joining the points (X_1, Y_1, Z_1), (X_2, Y_2, Z) is

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0.$$

Ex. 2. The points (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) are in line if and only if

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = 0.$$

Ex. 3. The line-equation of the point common to the lines (l_1, m_1, n_1), (l_2, m_2, n_2) is

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Ex 4. The lines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are concurrent if and only if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

(v) The parametric equations of a line and of a point.

(*a*) The discussion here is on the same lines as in section 2, part (iv).

Let (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) be any two distinct points in E_M. If (X, Y, Z) is any point on the line joining these, there exist l, m, n, not all zero, such that

$$\begin{aligned} lX + mY + nZ &= 0, \\ lX_1 + mY_1 + nZ_1 &= 0, \\ lX_2 + mY_2 + nZ_2 &= 0, \end{aligned}$$

and, therefore, such that

$$l(\lambda X_1 + \mu X_2) + m(\lambda Y_1 + \mu Y_2) + n(\lambda Z_1 + \mu Z_2) = 0,$$

for all λ, μ.

If $l \neq 0$ (the argument is similar if $m \neq 0$ or $n \neq 0$), we choose $\lambda : \mu$ so that

$$Y : Z = \lambda Y_1 + \mu Y_2 : \lambda Z_1 + \mu Z_2.$$

Then

$$lX(\lambda Z_1 + \mu Z_2) = lZ(\lambda X_1 + \mu X_2),$$

and therefore

$$X : Z = \lambda X_1 + \mu X_2 : \lambda Z_1 + \mu Z_2$$

Since X, Y, Z are not all zero, λ, μ cannot both be zero.

Thus the modified co-ordinates of every point on the line may be expressed in the form

$$X : Y : Z = \lambda X_1 + \mu X_2 : \lambda Y_1 + \mu Y_2 : \lambda Z_1 + \mu Z_2$$

with λ, μ not both zero

In these equations the ratio but not the absolute values of λ, μ is significant. The ordered pair of numbers (λ, μ) is called a pair of *homogeneous parameters* of the point (X, Y, Z), and the equations themselves are called *parametric equations of the line*.

The point (X_1, Y_1, Z_1) arises when $\lambda : \mu = 1 : 0$; the point (X_2, Y_2, Z_2) arises when $\lambda : \mu = 0 : 1$; and the inaccessible point on the line arises when $\lambda : \mu = -Z_2 : Z_1$.

We have proved incidentally that the points of the line may be put in $(1, 1)$ correspondence with all *bisets* $[\lambda, \mu]$, excluding $[0, 0]$, a biset being defined analogously to a terset.

(*b*) In E_M, the set of lines which pass through any given point is called a *pencil* with *vertex* at the point.

We may prove exactly as above that the co-ordinates of every

line in the pencil determined by the two lines (l_1, m_1, n_1), (l_2, m_2, n_2) may be expressed in the form

$$l : m : n = \lambda l_1 + \mu l_2 : \lambda m_1 + \mu m_2 : \lambda n_1 + \mu n_2,$$

and *vice versa*. These equations are called the *parametric equations of the vertex* or of the pencil.

Ex. 5 The point where the line $aX + bY + cZ = 0$ meets the line joining (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) has parameters given by

$$\lambda : \mu = -(aX_2 + bY_2 + cZ_2) : (aX_1 + bY_1 + cZ_1).$$

Ex. 6 The line common to the pencil $lA + mB + nC = 0$ and the pencil determined by (l_1, m_1, n_1), (l_2, m_2, n_2) has parameters given by

$$\lambda : \mu = -(l_2A + m_2B + n_2C) . (l_1A + m_1B + n_1C).$$

4. Groups and fields.

(i) Groups.—It is convenient at this stage to refer briefly to certain algebraic concepts which are of importance in algebraic geometry.

Let G denote a set of objects, called the *elements* of G, and let these be denoted by symbols a, b, c, .. If two symbols, say a, b, denote the same element we say that they are *equal* and indicate this by writing $a = b$. It is clear that equality is reflexive (that is: $a = b$ implies $b = a$) and transitive (that is: $a = b$ and $b = c$ together imply $a = c$).

Let A denote a rule by which with every ordered pair of elements a, b of G is associated a unique object denoted by $a \circ b$. The objects $a \circ b$ and $b \circ a$ may be different

G is said to be a *group* with respect to A if, for every a, b, c of G,

(1) $a \circ b$ is an element of G;

(2) $a \circ (b \circ c) = (a \circ b) \circ c$, where the brackets signify that the object associated with the pair of elements within the brackets is associated with the remaining element;

(3) there exist elements x, y of G such that

$$a \circ x = b, \quad y \circ a = b.$$

Ex. 1. Let G denote either the set of all (finite) real numbers or the set of all (finite) complex numbers and let A denote addition, so that $\circ$ means $+$. Then G is a group with respect to A.

Ex. 2. Let G be the set of transformations given by equations

of the form $x' = x + a$, where x', x, a are real numbers and the number a, called a parameter, is fixed for each transformation. Let T_a denote the element of G with parameter a. If A is defined by $T_a \circ T_b = T_{a+b}$, show that G is a group with respect to A.

The equation of T_a may be taken to represent a displacement, in a real euclidean plane, along the x-axis through a distance a. Then we are asserting that the set of displacements is a group with respect to the rule A.

Alternatively, we may interpret the numbers as the angular co-ordinates of lines lying in a given real euclidean plane and passing through a fixed point. T_a then represents a rotation in the positive sense through an angle a. These rotations form a group with respect to the rule A.

Ex. 3. Let G be the set of transformations of which a typical element $T(a, b, c, d)$ is given by

$$' = \frac{ax + b}{cx + d},$$

where the numbers involved are all either rational or real or complex, and $ad \neq bc$. Such transformations (which include those of *Ex.* 2 as a special case) are of fundamental importance in this book and will later be treated under the heading of *projective transformations.*

Let A be defined by the relation

$$T(a, b, c, d) \circ T(p, q, r, s) = T(pa + qc, pb + qd, ra + sc, rb + sd)$$

and show that G is a group with respect to A.

Ex. 4. A set of six transformations of the type specified in *Ex.* 3 is the set H

$$\begin{array}{ll} x' = x, & x' = 1/x, \\ x' = 1 - x, & x' = 1/(1 - x), \\ x' = x/(x - 1), & x' = (x - 1)/x. \end{array}$$

Show that H is a group (called a *subgroup* of G) with respect to the rule A of *Ex.* 3.

The expressions on the right hand sides of these equations will be encountered in the theory of *cross-ratio.*

Ex. 5. Let G be the set of all transformations of the form

$$\begin{aligned} x' &= \lambda x + \mu y + \nu, \\ y' &= \lambda' x + \mu' y + \nu', \end{aligned}$$

involving real numbers, where

$$\lambda^2 + \mu^2 = \lambda'^2 + \mu'^2 = 1, \; \lambda\lambda' + \mu\mu' = 0.$$

Denoting the transformation just written by $T(\lambda, \mu, \nu; \lambda', \mu', \nu')$, we define A by the relation

$$T(\lambda, \mu, \nu; \lambda', \mu', \nu') \circ T(\lambda_1, \mu_1, \nu_1; \lambda_1', \mu_1', \nu_1')$$
$$= T(\lambda\lambda_1 + \lambda'\mu_1, \mu\lambda_1 + \mu'\mu_1, \nu\lambda_1 + \nu'\mu_1 + \nu_1; \lambda\lambda_1' + \lambda'\mu_1', \mu\lambda_1' + \mu'\mu_1', \nu\lambda_1' + \nu'\mu_1' + \nu_1').$$

Show that G is a group with respect to A.

G is the set of equations representing the change from distance co-ordinates relative to one pair of rectangular axes to distance co-ordinates relative to another pair of rectangular axes in a euclidean plane.

Ex. 6. Let G be the set of all transformations of the form

$$X_0' : X_1' : \ldots : X_n' = \sum_{j=0}^{n} a_{0j}X_j : \sum_{j=0}^{n} a_{1j}X_j : \ldots : \sum_{j=0}^{n} a_{nj}X_j,$$

where the numbers involved are all real or all complex and

$$\begin{vmatrix} a_{00} & a_{01} \ldots a_{0n} \\ a_{10} & a_{11} \ldots a_{1n} \\ \ldots & \ldots \\ a_{n0} & a_{n1} \ldots a_{nn} \end{vmatrix} \neq 0.$$

Denoting the transformation just written by $T(a)$, we define A by the relation $T(a) \circ T(b) = T(c)$, where

$$c_{ik} = \sum_{j=0}^{n} a_{ij}b_{jk}, \quad i, k = 0, 1, \ldots, n.$$

Prove that G is a group with respect to A. It is called the *projective group*. If $n = 1$, the group is a restatement in terms of homogeneous co-ordinates of the group mentioned in *Ex.* 3. If $n = 2$, the group is that of the *projective collineations* of a modified euclidean or of a *projective* plane which form an important subject for later discussion. If n has any value greater than zero, the group is that of the projective collineations of a *projective space of* n *dimensions*.

(ii) Some elementary aspects of group theory.

(*a*) Let the set G be a group with respect to a rule A. The number n of elements of G may be finite or infinite; in the first case G is called a *finite group* and n is called its *order*, and in the second case G is called an *infinite group*.

A subset H of elements of G is called a *subgroup* of G if H is a group with respect to A.

(*b*) Let G, G′ be sets which are groups with respect to the rules A, A′ respectively; and let $\circ$ and $\wedge$ denote the corresponding symbols of association

A (1, 1) correspondence C between the elements of G, G′ such that, for every pair of elements a, b of G,

$$\mathsf{C}(a \circ b) = \mathsf{C}(a) \wedge \mathsf{C}(b),$$

where $\mathsf{C}(x)$ denotes the element of G′ corresponding to the element x of G, is called an *equivalence* or *simple-isomorphism* between the groups.

Equivalent groups are formally distinct but have the same group-properties.

Ex. 7. If $\mathsf{C}^{-1}(y')$ denotes the element of G corresponding to the element y' of G′, prove that, if G, G′ are equivalent,

$$\mathsf{C}^{-1}(a' \wedge b') = \mathsf{C}^{-1}(a') \circ \mathsf{C}^{-1}(b').$$

Ex. 8. Let G be the set of all real numbers and A be addition; and let G′ be the set of all real powers of a non-zero real number x and A′ be multiplication. Show that G, G′ are groups with respect to A, A′ respectively and that these groups are equivalent.

Ex. 9. The group of *Ex.* 3 is equivalent to the group of *Ex.* 6 in the case $n = 1$.

(*c*) If G′ is G and A′ is A, an equivalence of the group with itself is called an *automorphism*. Clearly every group is automorphic under the *identity automorphism* which consists in making every element correspond to itself interest arises when other automorphisms exist

Ex 10. Define a rule of association with respect to which the set of automorphisms of a given group is itself a group.

Ex 11. If G is the set of all rational numbers (positive, negative and zero) and A is addition, G is a group with respect to A. A (1, 1) correspondence may be defined by $\mathsf{C}(a) = na$, where a is any element of G and n is a given positive integer Show that C is an automorphism of the group.

(*d*) A group with the property that $a \circ b = b \circ a$, for all a, b, is called *commutative* or *abelian*. In such a case it is often convenient to replace the symbol $\circ$ by the symbol $+$ and to describe the rule A as *addition*; with such a convention the group is also called *additive*

Ex. 12. The group of *Ex.* 1 is commutative; the group of *Ex.* 3 is not commutative.

(*e*) We prove that every group possesses a unique element, called the *identity element* and denoted by e, such that, for every element a,

$$a \circ e = e \circ a = a.$$

By the third characteristic property of a group (see part (i) of this section) there exist elements e, f such that, given a,

$$a \circ e = f \circ a = a.$$

For the same reason, given b, there exist x, y such that

$$b = a \circ x = y \circ a.$$

Therefore

$$b \circ e = (y \circ a) \circ e = y \circ (a \circ e) = y \circ a = b,$$

and, similarly, $f \circ b = b$.

Taking $b = f$, $b = e$ in turn we infer $f = f \circ e = e$. Thus an element e exists such that, for all b, $b \circ e = e \circ b = b$. This element is unique, for if e' has the same property, we obtain, by taking $b = e'$, $b = e$ in turn, $e \circ e' = e' \circ e = e = e'$.

If the group is abelian and the additive notation is used, it is customary to denote the identity element by 0 and to call it the *zero element*; then, for every element a, we have

$$a + 0 = 0 + a = a.$$

Ex 13. In *Exs*. 1, 2, 3, 4, 5 and 6 the identity elements are respectively

(i) The number 0;

(ii), (iii), (iv) The transformation $x' = x$.

(v) The transformation $x' = x$, $y' = y$.

(vi) The transformation

$$X_1' : X_2' : \ldots : X_n' = X_1 . X_2 : \ldots : X_n.$$

(*f*) We prove that there is associated with every element a of a group a unique element, called the *inverse* of a and denoted by a^{-1}, such that

$$a \circ a^{-1} = a^{-1} \circ a = e.$$

By the third characterising property of a group, there exists an element a^{-1} such that $a \circ a^{-1} = e$ Then

$$a^{-1} \circ (a \circ a^{-1}) = a^{-1} \circ e = a^{-1},$$

whence

$$(a^{-1} \circ a) \circ a^{-1} = a^{-1}.$$

Therefore $$a^{-1} \circ a = e.$$

The uniqueness of a^{-1} follows from the fact that the relation $a \circ b = e$ implies

$$b = e \circ b = (a^{-1} \circ a) \circ b = a^{-1} \circ (a \circ b) = a^{-1} \circ e = a^{-1}.$$

In the case of an additive group, we use the notation $-a$ to denote the inverse of a; then we have $a + (-a) = (-a) + a = 0$. The notation is abbreviated by writing $a - b$ for $a + (-b)$ or $(-b) + a$.

Ex. 14. If $a \circ x = b$ and $y \circ a = b$, then $x = a^{-1} \circ b$, $y = b \circ a^{-1}$.
Ex. 15. Find the inverse of the elements named in *Exs.* 2–6

(iii) Fields.—Let the set F be an additive (abelian) group with respect to a rule A and let there be a second rule M, called *multiplication*, for associating the pairs of elements in such a way that, for all a, b, c of F,

(1) M associates with a, b a unique element of F called the *product* of b by a and denoted by ab;
(2) $ab = ba$;
(3) $a(bc) = (ab)c$,
(4) $a(b + c) = ab + ac$;
(5) given a, different from zero, and b there exists an element of F such that $ax = b$.

Then F is said to be a *field* with respect to the rules A, M

The non-zero elements of F clearly form a group with respect to M. The identity element of this group is called the *unit element* of F and denoted by 1. Thus, for all a in F, we have $a1 = 1a = a$. Further, the inverse of any element a of the group just mentioned is called the *reciprocal* of a and denoted by a^{-1}; we have $aa^{-1} = a^{-1}a = 1$.

Ex. 16. The sets of all rational numbers, of all real numbers, and of all complex numbers are fields with respect to the ordinary rules of addition and multiplication.

Ex. 17. Let F be the set of real number residues modulo p, a prime number, and let A, M respectively denote addition and multiplication modulo p; then F is a field with respect to A, M

(iv) Some elementary aspects of field theory.

(*a*) We prove some important properties of the elements of a field.

First, for all a, $a0 = 0a = 0$. Since $0 + 0 = 0$, we have $a0 + a0 = a(0 + 0) = a0$; moreover 0 is the unique solution of $a0 + x = a0$; hence $0 = a0$. Similarly $0 = 0a$.

Next, for all a, b, $a(-b) = (-ab) = -(ab)$. We have $0 = a0 = a(b - b) = ab + a(-b)$; moreover $-(ab)$ is the unique solution of $0 = ab + x$; hence $a(-b) = -(ab)$. Similarly $(-a)b = -(ab)$. It follows that $(-a)(-b) = ab$.

If $ab = 0$ then at least one of a, b is 0. Let us suppose the conclusion to be false; then $a \neq 0$, $b \neq 0$. Since $a \neq 0$, a^{-1}

exists and is non-zero and we have $b = 1b = (a^{-1}a)b = a^{-1}(ab) = a^{-1}0 = 0$, which contradicts the supposition. The conclusion is therefore true.

Ex. 18. Denoting xy^{-1} by x/y, for all x and for all $y \neq 0$, prove that

$$(a/b)(b/a) = 1,$$
$$c/a + d/b = (bc + ad)/(ab),$$
$$(ka)/(kb) = a/b,$$

where a, b, k are non-zero.

(*b*) Let C be a (1, 1) correspondence between the elements of two fields F, F' such that (with the notation of part (ii) (*b*) of this section) for every pair of elements a, b of F,

$$\mathsf{C}(a + b) = \mathsf{C}(a) + \mathsf{C}(b),$$
$$\mathsf{C}(ab) = \mathsf{C}(a)\mathsf{C}(b),$$

where, on the left, addition and multiplication are defined by rules A, M respectively for F, and, on the right, are defined by rules A', M' respectively for F'. Then C is called an *equivalence* between the two fields, which are themselves called *equivalent*.

Equivalent fields are formally distinct but have the same field-properties.

Ex. 19. Prove that

$$\mathsf{C}^{-1}(a' + b') = \mathsf{C}^{-1}(a') + \mathsf{C}^{-1}(b'),$$
$$\mathsf{C}^{-1}(a'b') = \mathsf{C}^{-1}(a')\mathsf{C}^{-1}(b'),$$

where a', b' are any two elements of F', and, on the left, addition and multiplication are by A', M', and, on the right, by A, M respectively.

If $0'$, $1'$ denote respectively the zero and unit elements of F', prove also that

$$\mathsf{C}(0) = 0', \quad \mathsf{C}(1) = 1',$$
$$\mathsf{C}^{-1}(0') = 0, \quad \mathsf{C}^{-1}(1') = 1.$$

(*c*) A *subfield* K of a field F is a subset of F which is a field with respect to the rules A, M of F. The zero and unit elements of F belong to every subfield. We say also that F is an *extension* of K.

Let F', K be fields, with no elements in common, such that F' contains a subfield K' equivalent to K. We prove that there exists a field F, equivalent to F', which is an extension of K.

Let A', M' be respectively the rules for addition and multiplication for F' and A_1, M_1, be the corresponding rules for K.

The set F is defined to be the set of elements of K together with the set of elements of F' which are not in K'.

Addition A and multiplication M are defined for F thus: let p, q be any two elements of F; if p, q both belong to K, define $p + q$ and pq by A_1, M_1; if p, q both belong to F′ but not to K′, define $p + q$ and pq by A′, M′; if p belongs to K and q does not, let p' in K′ correspond to p in K and define $p + q$ to be $p' + q$ relative to A′ and pq to be $p'q$ relative to M′.

Evidently F is a field with respect to A, M and is equivalent to F′.

(*d*) The field F′ of (finite) complex numbers is usually defined to be the set of all ordered pairs (a, b) of real numbers, with addition defined by

$$(a, b) + (c, d) = (a + c, b + d),$$

where $a + c, b + d$ are ordinary arithmetical sums, and multiplication is defined by

$$(a, b)(c, d) = (ac - bd, ad + bc),$$

where ac, bd, ad, bc are ordinary arithmetical products. The zero element is $(0, 0)$ and the unit element is $(1, 0)$

The subfield K′ of all complex numbers of the form $(a, 0)$ is equivalent to the field K of all real numbers. We may construct, as above, an extension F of K which is equivalent to F′. Since F, F′ are equivalent, it is not uncommon to find either F or F′ referred to as the field of (finite) complex numbers.

(v) The number ∞.

(*a*) Let F be the field of all complex numbers defined in part (iv) (*d*) of this section. Let F_∞ denote the set whose elements are all those of F together with one other element, called *infinity* and denoted by ∞.

Rules of addition A_∞ and of multiplication M_∞ are defined for F_∞ as follows. If a and b are both in F, $a + b$ and ab are defined as for F For every a in F we define $a + \infty = \infty = \infty + a$ and, except for $a = 0$, $a\infty = \infty = \infty a$. We define $\infty + \infty$, 0∞ and $\infty 0$ to be *indeterminates*, that is, symbols each capable of denoting any element of F_∞; and we define $\infty\infty = \infty$.

In making these conventions it is necessary to give up some familiar properties of the relation $=$. A statement that $a = b$ in F_∞ means (as in F) that there is an element in F_∞ representable both by a and by b. But none of the statements (i) $a + c = b + c$, (ii) $ac = bc$ with $c \neq 0$, (iii) $a = c$ together with $b = c$, necessarily implies that $a = b$, unless a, b, c all belong to F, for example, (i) $1 + \infty = 2 + \infty$, (ii) $1\infty = 2\infty$, (iii) $1 = 0\infty$, $2 = 0\infty$.

With a, b in F_∞, let $a - b$ denote any element x such that $a = b + x$, and a/b any element y such that $a = by$. Then $0 - \infty = \infty = (-1)\infty$; and $\infty - \infty$ is indeterminate. Also $1/0 = \infty$, $1/\infty = 0$, and ∞/∞ and $0/0$ are also indeterminate.

If a, b, c belong to F and $a \neq 0$, $b \neq c$, the equations $0x + a = 0$ and $ax + b = ax + c$ are each satisfied uniquely in F_∞ by $x = \infty$. The equation $0x^2 + ax + b = 0$ is satisfied by $x = -b/a$ and by $x = \infty$. The equation $0x^2 + 0x + a = 0$ is satisfied only by $x = \infty$; we agree to say that ∞ is a double root of this equation, to conform with the convention that 0 is a double root of $0 + 0x + ax^2 = 0$.

The set F_∞ is *not* a field with respect to A_∞, M_∞. With respect to these rules we call F_∞ the *modified system of complex numbers.* The set consisting of ∞ and the real numbers of F is the *modified system of real numbers.*

(*b*) Let $f(z)$ be a single-valued function defined over a domain in F, the values of $f(z)$ being in F; and let a be in F. If u exists in F so that $|f(z) - u|$ is made arbitrarily small by taking any z in the domain for which the positive number $|z - a|$ is small enough, we define $\lim_{z \to a} f(z) = u$. If $|f(z)|$ is made arbitrarily large by such z, we define $\lim_{z \to a} f(z) = \infty$. Finally, we define $\lim_{z \to \infty} f(z)$ to be $\lim_{w \to 0} f(1/w)$ if this exists.

(*c*) The extension of F by adjoining the element ∞ allows us to extend, as we shall need to do, the range of definition of the function $f(z) \equiv (az + b)/(cz + d)$, where a, b, c, d are in F and $ad \neq bc$.

If $cv + d = 0$, we define $f(v) = \lim_{z \to v} f(z) = \infty$; and we define $f(\infty) = \lim_{z \to \infty} f(z) = a/c$ $(= \infty$ if $c = 0)$ The function $f(z)$ is now defined all over F_∞ and has the property that $\lim_{z \to w} f(z) = f(w)$ for all w in F_∞.

Similarly, the symbol $\{a, b; c, d\}$, which denotes $\left(\dfrac{a - c}{c - b}\right) \Big/ \left(\dfrac{a - d}{d - b}\right)$, is defined for any set of unequal numbers a, b, c, d in F and when $a = c$.

When $b = c$, we define $\{a, b; b, d\} = \lim_{x \to b} \{a, b; x, d\} = \lim_{x \to b} \{a, x; b, d\} = \infty$, and similarly for other equalities among a, b, c, d. (It will be noticed that $\{a, a; a, d\}$, for example, is indeterminate.) And then we define $\{\infty, b; c, d\} = \lim_{x \to \infty} \{x, b; c, d\} = (d - b)/(c - b)$, and $\{\infty, b; \infty, d\} = \lim_{x \to \infty} \{x, b, x, d\} = 0$; and so on.

Again, if we have parametric equations of a line in E_M in the form (section 3(v))

$$X : Y \quad Z = \lambda X_1 + \mu X_2 : \lambda Y_1 + \mu Y_2 : \lambda Z_1 + \mu Z_2,$$

with λ, μ in F and not both 0, we now see that, by the substitution $\lambda/\mu = \theta$, it is possible to express the equations in the form

$$X : Y : Z = \theta X_1 + X_2 : \theta Y_1 + Y_2 : \theta Z_1 + Z_2,$$

where θ ranges through F_∞, and *vice versa*, provided that we interpret the latter equations when $\theta = \infty$ as meaning the same as the first when $\mu = 0$; that is, the value of θ assigned to (X_1, Y_1, Z_1) is ∞. We call θ a *non-homogeneous parameter*.

Lastly, in regard to tersets, if a, b, c, d, e, f are in F and a, c, e are not all 0, we define

$$\lim_{z\to\infty} [az + b, cz + d, ez + f] = [a, c, e].$$

5. Terminology.

(i) Tangential co-ordinates. Points at infinity.—The historical development of geometry has produced various names for what are essentially the same things Thus equations involving l, m, n arose from the consideration of what condition is satisfied by the co-ordinates of a line in order that it should touch a given curve; and, in this way, these equations came to be known as *tangential equations* and l, m, n as *tangential co-ordinates*.

Inaccessible points are still often called *points-at-infinity*. This name, which is misleading without explanation, arose through the following considerations.

Let E denote a real or complex euclidean plane and E_M the covering modified plane. Let L denote a line in E and L_M denote the covering line in E_M. Relative to any algebraic system of co-ordinates in E, let L have the equation $lx + my + n = 0$. Then a set of modified co-ordinates of any point P on L may be taken in the form $(m\xi, -(l\xi + n), m)$. The point P has no limiting position on L when $\xi \longrightarrow \infty$: on the other hand the terset $[m\xi, -(l\xi + n), m]$ has a limit, namely $[m, -l, 0]$, which is the terset for the inaccessible point P_∞ on L_M. We can, if we wish, extend the geometrical significance of the word "limit" by defining P_∞ to be the *limit in* E_M of P as $\xi \longrightarrow \infty$. And we may, if we wish, carry a suggestion of this limit for ξ by providing for P_∞ the alternative name of *point-at-infinity* on L_M. In accordance with this nomenclature we should then give the inaccessible line the alternative name of *line-at-infinity*. We prefer, however, to use the term "inaccessible"; and would warn the reader against imagining that P_∞ is some very distant point in E : it is not.

(ii) Extension of the meanings of familiar terms. Circular points.—Names associated with configurations in a real euclidean plane are often used to describe related configurations in a complex or modified euclidean plane. We have already met examples of this usage in regard to the words "point" and "line."

As a further example, let us consider a circle in a real euclidean plane E_R. With rectangular axes and distance co-ordinates the circle has an equation of the form

$$(x - a)^2 + (y - b)^2 = r^2.$$

The modified co-ordinates of the points of the circle satisfy the equation

$$(X - aZ)^2 + (Y - bZ)^2 = r^2Z^2.$$

This equation defines a locus in the covering modified complex euclidean plane E_M which we also call a *circle.*

It is important to remark here that all circles in E_M meet the inaccessible line in the same pair of distinct inaccessible points given by

$$X^2 + Y^2 = 0, Z = 0.$$

These points therefore have modified co-ordinates $(1, i, 0)$, $(1, -i, 0)$, which may easily be shown to be the same for all pairs of rectangular axes. The points are called the *circular* or *absolute points* of E_M. It will appear later that the circular points are of fundamental importance in relation to the metrical geometry of E_R, when we regard E_R as embedded in E_M.

A circle for which the *radius* r is zero is called a *point-circle.* In E_R the equation of such a circle represents a single point, namely (a, b) with the above notation. In E_M, however, the equation represents two lines, namely those with the equations

$$Y - bZ = i(X - aZ), \; Y - bZ = - i(X - aZ),$$

which join the point $(a, b, 1)$ to each of the circular points. These lines are called the *isotropic lines* through the point. The isotropic lines in E_M form two pencils with vertices at the circular points.

As in the case of a complex euclidean plane, we say that the two lines in E_M with modified distance-co-ordinates (l_1, m_1, n_1), (l_2, m_2, n_2) are *perpendicular* when $l_1l_2 + m_1m_2 = 0$ Every isotropic line is perpendicular to itself.

(iii) Algebraic curves and envelopes.

(*a*) In a modified complex euclidean plane the set of points whose co-ordinates satisfy an equation of the form $\phi_n(X, Y, Z) = 0$, where ϕ_n is a homogeneous polynomial of degree n in X, Y, Z and has real or complex coefficients, is called an *algebraic curve of order* n. A line is thus an algebraic curve of order 1. Algebraic curves of orders 2, 3, 4, . . . are known as *conics, cubics, quartics,* . . . .

If the polynomial ϕ_n is reducible, in the field of complex numbers, the curve is called *reducible*; otherwise it is called *irreducible.* If an irreducible factor ϕ_m occurs r times in ϕ_n, the irreducible curve $\phi_m = 0$ is said to *count* r *times as part of the curve* $\phi_n = 0$.

Thus a reducible conic consists of a pair of distinct lines or of a single line counted twice. A reducible cubic consists of a line and an irreducible conic, or of three distinct lines, or of one line, counted twice, and another line, or of one line, counted three times.

In particular, the name *circle* is used further to describe any conic, reducible or irreducible, which passes through both circular points.

(*b*) Algebraic curves are similarly defined for any of the other types of plane which we have considered. In the case of an unmodified plane, the defining polynomial is, however, a non-homogeneous polynomial $\psi_n(x, y)$ of degree n in the variables x, y. This may be expressed in the form $\psi_n(x, y) \equiv \phi_n(x, y, 1)$, where $\phi_n(x, y, z)$ is homogeneous and of degree n in x, y, z.

The reducibility of an algebraic curve is defined relative to the field of numbers appropriate to the plane considered. In this connection it should be pointed out explicitly that while the polynomial $\phi_n(X, Y, Z)$ may be reducible relative to the field of complex numbers, it may not be reducible relative to the field of real numbers. We have already had an example of such a polynomial, namely $X^2 + Y^2$; in connection with this the conic $X^2 + Y^2 = 0$ is reducible in the modified complex euclidean plane and consists of two isotropic lines but the conic is irreducible in the modified real euclidean plane and consists of the single point (0, 0, 1)

(*c*) In a modified complex euclidean plane, the set of lines whose co-ordinates satisfy an equation of the form $\phi_n(l, m, n) = 0$, where ϕ_n is a homogeneous polynomial of degree n in l, m, n and has real or complex coefficients, is called an *algebraic envelope of class* n. Thus a pencil of lines is an algebraic envelope of class 1. Algebraic envelopes of classes 2, 3, 4, . . . are known as *conic-envelopes*, *cubic-envelopes* or *class-cubics*, *quartic-envelopes* or *class-quartics*, . . .

Reducible envelopes are defined in the same way as reducible curves. Thus a reducible conic-envelope consists of two distinct pencils or of one pencil counted twice.

The definition may also be applied, like that of an algebraic curve, to the other types of plane.

6. The principle of duality.

In a modified euclidean plane, every statement regarding the incidences in a configuration of points, lines, curves, envelopes amounts to a statement regarding the algebraic relations between certain sets of point co-ordinates (X, Y, Z), sets of line co-ordinates (l, m, n), polynomials $\phi(X, Y, Z)$, polynomials $\psi(l, m, n)$.

If we now choose to regard each triad (X, Y, Z) as a set of co-ordinates of some *line*, and each triad (l, m, n) as a set of co-ordinates of some *point*, these algebraic relations amount to a statement regarding the incidences in a configuration of lines, points, envelopes, curves. Thus every theorem of the first kind is associated with a theorem of the second kind; each of these theorems is called a *dual* of the other, and they are characterised by the fact that, except for their dual phrases, the *same* algebra proves both theorems.

To proceed from a theorem to its dual, all that is necessary is to substitute for certain phrases their dual phrases, according to the following scheme, in which the dual of any phrase is the one opposite.

Table of dual phrases.

Point.	Line.
Points of a line.	Lines of a pencil.
Line joining two points.	Point common to two lines.
Collinear points.	Concurrent lines.
Curve.	Envelope.

Some examples of dual theorems have already occurred in the text. Two important examples are the following.

There are precisely n points of an algebraic curve of order n on any line.

There are precisely n lines of an algebraic envelope of class n in any pencil.

Each of these theorems is an expression of the fact that, if ϕ_n is a homogeneous polynomial in the co-ordinates of degree n, the equation in λ/μ

$$\phi_n(a_1\lambda + a_2\mu, b_1\lambda + b_2\mu, c_1\lambda + c_2\mu) = 0,$$

has precisely n roots.

If A, B, C are three points of one line, and A', B', C' are three points of another line, then the three intersections $BC'.B'C$, $CA'.C'A$, $AB'.A'B$ are collinear.

If A, B, C are three lines of one pencil, and A′, B′, C′ are three lines of another pencil, then the joins of B . C′ to B′ . C, of C . A′ to C′ . A, of A . B′ to A′ . B are concurrent

The theorem on the left is well known as due to Pappus. A proof is given in section 12 (ii).

A triangle is a figure of three points and the lines joining these in pairs, the dual of a triangle is a triangle.

7. Review of Chapter I.

It is worth while to pause here and remark on the general nature of what we have done so far.

Foremost is the fact that we are applying algebra to the study of geometry. It is therefore natural to stress most those aspects of geometry which are represented by simple algebraic processes and also to seek to interpret geometrically any interesting piece of algebra.

The chapter is essentially devoted to laying the foundations on which to build a theory free, as far as possible, from statements of exception to generality.

In section 1 we have considered some aspects of the geometry of a real euclidean plane which are both of wide interest and of basic application later.

In section 2 we extended the geometrical domain of discussion with a view to the interpretation of complex numbers; ultimately this permits us to make the important generalisation that two algebraic curves, of orders m and n, meet in mn points, when these are counted in a specified way.

In section 3 we extended the geometrical domain further in order to avoid statements of exception in regard to parallelism and we introduced homogeneous co-ordinates, these being naturally adaptable to the new domain.

In section 4 it was convenient to present the elementary ideas connected with the important algebraic concepts of group and field. The idea of group will frequently arise in later discussion where we shall find that we have to deal with various types of transformation, those of any one type sometimes forming a group. And we shall study the *invariants* of such a group, namely, the entities, numerical or geometrical, which are unchanged by effecting any of the transformations of the group. For example, at the end of section 1 we saw that distance is an invariant of the group of transformations defining the change of distance co-ordinates from one pair of rectangular axes to another.

In dealing with a real or complex euclidean plane, it is important to appreciate that we are concerned with numbers drawn from the field of real numbers in the first case and from the field of complex numbers in the second case. In dealing with a modified plane, the same statement applies so long as we restrict ourselves to homogeneous co-ordinates, but here, if we wish (and we shall do so) to use non-homogeneous co-ordinates or parameters, it becomes necessary to employ the number ∞ in addition to the numbers of the field.

We are not obliged, from an algebraic point of view, to work relative to the particular fields of real and of complex numbers. A new kind of geometry arises if we work relative to a different field, as, for example, the so-called *finite geometry*, mentioned later in this book, which is associated with the field of residues modulo p, a prime number.

CHAPTER II

PROJECTIVE TRANSFORMATIONS. LINEAR GEOMETRY

8. Bonds.

(*a*) We consider here a modified real or complex euclidean plane, bearing in mind the appropriate number-system.

Let A, B be two given points with respective triads of co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2) in an assigned co-ordinate system. (There is no longer any need to distinguish homogeneous co-ordinates by capital letters.) Every point P on the line AB may be assigned a triad of co-ordinates in the form $(\lambda x_1 + \mu x_2.$ $\lambda y_1 + \mu y_2, \lambda z_1 + \mu z_2)$. We symbolise the relation between these three triads of co-ordinates by writing

and call this relation a *bond*; other writers say *syzygy*.

(Note that we are now proposing to use a capital italic letter to denote either a point or a *specific* triad of co-ordinates of a point, and to put the symbol $+$ to a new use; the context will always make clear which meaning is intended.)

More generally, let $P_1, P_2, \ldots, P_n$ be any set of n points in the plane with the respective triads of co-ordinates (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$. The triad

$$(a_1x_1 + a_2x_2 + \ldots + a_nx_n, \quad a_1y_1 + a_2y_2 + \ldots + a_ny_n, \quad a_1z_1 + a_2z_2 + \ldots + a_nz_n)$$

is either the triad $(0, 0, 0)$ or else is a triad of co-ordinates for some point P. In the first case we write

$$a_1P_1 + a_2P_2 + \ldots + a_nP_n \equiv 0,$$

calling this relation a *nul-bond*; and in the second case we write

$$a_1P_1 + a_2P_2 + \ldots + a_nP_n \equiv P,$$

calling this relation also a *bond*.

For brevity, we write $A - B$ for $A + (-1)B$. Clearly, if $A \equiv B$, then $A - B \equiv 0$, and *vice versa*. And if $A + B \equiv C$, then $A \equiv C - B$, and *vice versa*.

Three points A, B, C are in line if and only if they are in bond. For, if the points are in line, there exists a bond $C \equiv \lambda A + \mu B$, whence $\lambda A + \mu B - C \equiv 0$. Conversely, if $\lambda A + \mu B + \nu C \equiv 0$,

we have $C \equiv -(\lambda/\nu)A - (\mu/\nu)B$, showing at once that C is on the line AB. This is a most useful result.

Ex. 1. Any four points are in bond.

Ex. 2. If $P \equiv bB - cC$, $Q \equiv cC - aA$, $R = aA - bB$, then P, Q, R are in line.

(*b*) The concept of bond may also be applied to the triads of co-ordinates representing lines in an obvious manner.

A most useful result is that three lines are concurrent if and only if they are in bond.

9. Projectivities.

(i) Bilinear equations.

(*a*) In this part of the section we draw numbers from the modified system of real or of complex numbers.

An equation of the form

$$a\lambda\mu + b\lambda + c\mu + d = 0,$$

where the coefficients a, b, c, d are finite and not all zero, is called *bilinear* since it is linear in each of λ, μ.

To every value of λ corresponds a value of μ given by

$$\mu = -\frac{b\lambda + d}{a\lambda + c}.$$

This value of μ varies with λ, and is different for every value of λ, unless $b : d = a : c$, when the value is fixed except when $\lambda = -c/a$, in which case the value of μ is arbitrary. Similar remarks apply when λ is expressed in terms of μ.

The bilinear equation is called *singular* when $ad = bc$, *non-singular* when $ad \neq bc$. Unless it is explicitly stated otherwise, we shall consider only non-singular bilinear equations.

Ex. 1. If $a \neq 0$, then $\mu = -b/a$ when $\lambda = \infty$, and $\lambda = -c/a$ when $\mu = \infty$. If $a = 0$, then $\lambda = \infty$ when $\mu = \infty$.

(*b*) The homogeneous equation in θ, ϕ and θ', ϕ'

$$\theta' : \phi' = -(b\theta + d\phi) : (a\theta + c\phi)$$

where the variables range through the field of real or complex numbers, is equivalent to the bilinear equation written above when we put

$$\lambda = \theta/\phi, \mu = \theta'/\phi'.$$

We may call this equation the homogeneous form of the non-homogeneous bilinear equation.

(*c*) Let $$p\xi\eta + q\xi + r\eta + s = 0$$

be another non-singular bilinear equation, so that $ps \neq qr$. When $\xi = \lambda$, let $\eta = \lambda'$; and when $\xi = \mu$, let $\eta = \mu'$. We prove that λ', μ' are connected by a non-singular bilinear equation, which is said to be the *transform* of $a\lambda\mu + b\lambda + c\mu + d = 0$ by $p\xi\eta + q\xi + r\eta + s = 0$.

We have

$$\lambda = -\frac{r\lambda' + s}{p\lambda' + q}, \quad \mu = -\frac{r\mu' + s}{p\mu' + q};$$

therefore

$$a(r\lambda' + s)(r\mu' + s) - b(r\lambda' + s)(p\mu' + q)$$
$$- c(p\lambda' + q)(r\mu' + s) + d(p\lambda' + q)(p\mu' + q) = 0;$$

that is $$A\lambda'\mu' + B\lambda' + C\mu' + D = 0$$
where

$$\begin{aligned} AD - BC &= (ar^2 - bpr - cpr + dp^2)(as^2 - bqs - cqs + dq^2) \\ &\qquad - (ars - bqr - cps + dpq)(ars - bps - cqr + dpq) \\ &= (ad - bc)(ps - qr)^2 \\ &\neq 0 \end{aligned}$$

(ii) Projectivity on a line.

(*a*) We consider here a modified real or complex euclidean plane.

Let A, B be two points with co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively. Every point on the line AB has a triad of co-ordinates of the form $(\theta x_1 + x_2, \theta y_1 + y_2, \theta z_1 + z_2)$, where θ is a non-homogeneous parameter, there is one value of θ to every point and *vice versa*.

Let $$a\lambda\mu + b\lambda + c\mu + d = 0$$
be a non-singular bilinear equation in the variables λ, μ.

Then the point P, of parameter θ_P, may be said to be *transformed*, by means of the bilinear equation, into the point Q, of parameter θ_Q, when the bilinear equation is satisfied by $\lambda = \theta_P$, $\mu = \theta_Q$. Every point on AB is transformed in this way into just one point on the line and, conversely, every point on AB is the transform of just one point on the line. The (1, 1) transformation of the set of points on the line into itself thus established is called a *projective transformation* or *projectivity*. Denoting the transformation by the symbol T, we indicate the relation between P and Q by writing $T(P) = Q$.

Further, the point P may be said to be *transformed in the sense inverse to* T, by means of the bilinear equation, into the point R, of parameter θ_R, when the bilinear equation is satisfied by $\mu = \theta_P$, $\lambda = \theta_R$. Another projective transformation is thus established which we call the *inverse transformation* to T and denote by T^{-1}. The relation between P and R is indicated by writing $T^{-1}(P) = R$.

Clearly, T is the inverse transformation to T^{-1}; that is to say $(T^{-1})^{-1}$ is T.

The parameters of $T(P)$ and of $T^{-1}(P)$ are respectively $-(b\theta_P + d)/(a\theta_P + c)$ and $-(c\theta_P + d)/(a\theta_P + b)$.

The definition of projectivity appears at first sight to depend on the pair of points A, B; we prove that this is not the case.

Let C, D be any two points, with specified co-ordinates, on the line AB. There exist bonds

$$A \equiv \alpha_1 C + \alpha_2 D, \quad B \equiv \beta_1 C + \beta_2 D,$$

with $\alpha_1 : \alpha_2 \neq \beta_1 : \beta_2$. Hence

$$P \equiv \theta_P A + B \equiv (\theta_P\alpha_1 + \beta_1)C + (\theta_P\alpha_2 + \beta_2)D,$$
$$Q \equiv \theta_Q A + B \equiv (\theta_Q\alpha_1 + \beta_1)C + (\theta_Q\alpha_2 + \beta_2)D.$$

If, therefore, the symbols $\theta_P'C + D$, $\theta_Q'C + D$ also represent the points P, Q respectively, we have

$$\theta_P'(\theta_P\alpha_2 + \beta_2) = \theta_P\alpha_1 + \beta_1,$$
$$\theta_Q'(\theta_Q\alpha_2 + \beta_2) = \theta_Q\alpha_1 + \beta_1.$$

The bilinear equation

$$\eta(\xi\alpha_2 + \beta_2) = \xi\alpha_1 + \beta_1$$

is non-singular; hence, by part (i) (c) of this section, θ_P', θ_Q' are connected by a non-singular bilinear equation

The projectivity T is thus expressed by a non-singular bilinear equation no matter which two points are taken to determine a parameterisation on the line. Moreover, by taking C at A and D at B, we see that T is expressed by a non-singular bilinear equation no matter what triads of co-ordinates, amongst those possible, are taken for the two points chosen to determine the parameterisation.

(b) Since a bilinear equation is determined by the ratios of three of the coefficients to the fourth, it follows that there is a unique projectivity in which three given points, say with parameters λ_1, λ_2, λ_3, correspond to three given points, say with parameters μ_1, μ_2, μ_3 respectively. One form of the equation of this projectivity is obviously

$$\begin{vmatrix} \lambda\mu & \lambda & \mu & 1 \\ \lambda_1\mu_1 & \lambda_1 & \mu_1 & 1 \\ \lambda_2\mu_2 & \lambda_2 & \mu_2 & 1 \\ \lambda_3\mu_3 & \lambda_3 & \mu_3 & 1 \end{vmatrix} = 0.$$

A more useful form of the equation is obtained as follows. A bilinear equation in which λ_1, λ_2 correspond to μ_1, μ_2 respectively is clearly of the form

$$A(\lambda_1 - \lambda)(\mu - \mu_2) = B(\lambda - \lambda_2)(\mu_1 - \mu);$$

it makes λ_3 correspond to μ_3 if and only if

$$A(\lambda_1 - \lambda_3)(\mu_3 - \mu_2) = B(\lambda_3 - \lambda_2)(\mu_1 - \mu_3).$$

The required equation, obtained by eliminating A, B, is therefore

$$\frac{(\lambda_1 - \lambda_3)(\lambda - \lambda_2)}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda)} = \frac{(\mu_1 - \mu_3)(\mu - \mu_2)}{(\mu_3 - \mu_2)(\mu_1 - \mu)}.$$

It is often useful to take μ_1, μ_2, μ_3 to be $0, \infty, 1$ respectively; then the equation becomes

$$\frac{(\lambda_1 - \lambda_3)(\lambda - \lambda_2)}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda)} = \frac{1}{\mu}.$$

(iii) The group property of projectivities.

(*a*) A projectivity has been defined by means of a bilinear equation connecting two non-homogeneous parameters. It is more convenient here to use homogeneous parameters, all with reference to one specified pair of base points, and to regard a projectivity A as defined equivalently by an equation

$$\theta' : \phi' = a_{11}\theta + a_{12}\phi : a_{21}\theta + a_{22}\phi.$$

The point (θ, ϕ) is transformed by A into the point (θ', ϕ'); we indicate this by writing $(\theta', \phi') = A(\theta, \phi)$. The condition of non-singularity is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

Let B be the projectivity defined by

$$\theta' : \phi' = b_{11}\theta + b_{12}\phi : b_{21}\theta + b_{22}\phi.$$

Let $\quad A(\xi, \eta) = (\xi', \eta'), \quad B(\xi', \eta') = (\xi'', \eta'');$

then

$$\xi'' : \eta'' = (b_{11}a_{11} + b_{12}a_{21})\xi + (b_{11}a_{12} + b_{12}a_{22})\eta$$
$$: (b_{21}a_{11} + b_{22}a_{21})\xi + (b_{21}a_{12} + b_{22}a_{22})\eta.$$

Thus (ξ, η) is transformed into (ξ'', η'') by the transformation C defined by the equation

$$\theta' : \phi' = c_{11}\theta + c_{12}\phi : c_{21}\theta + c_{22}\phi$$

where

$$c_{11} = b_{11}a_{11} + b_{12}a_{21}, \quad c_{12} = b_{11}a_{12} + b_{12}a_{22},$$
$$c_{21} = b_{21}a_{11} + b_{22}a_{21}, \quad c_{22} = b_{21}a_{12} + b_{22}a_{22}$$

and

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
$$\neq 0.$$

Thus C is a projective transformation. It is called the *product* of A by B and is denoted by BA.

It is to be observed that the matrix of the coefficients of C is the product (in the sense of algebra) of the matrix of A by the matrix of B. Note the order of the letters in BA.

The elements of a matrix of coefficients may all be multiplied by any common multiplier ($\neq 0, \infty$) without altering the projectivity.

(b) We now prove that, with respect to the rule of association just defined, the set of all the projectivities on the given line is a group.

First, the rule ensures that the product of two projectivities is a projectivity.

The second group property to be demonstrated is that, for any three projectivities A, B, D,

$$A(BD) = (AB)D.$$

Let the matrix of D be

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Then it is easily verified that both $A(BD)$ and $(AB)D$ have the matrix

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

where

$$\begin{aligned} e_{ij} &= a_{i1}(b_{11}d_{1j} + b_{12}d_{2j}) + a_{i2}(b_{21}d_{1j} + b_{22}d_{2j}) \\ &= (a_{i1}b_{11} + a_{i2}b_{21})d_{1j} + (a_{i1}b_{12} + a_{i2}b_{22})d_{2j}. \end{aligned}$$

The third group property to be shown is the existence of a projectivity X such that $AX = B$, that is to say that x_{11}, x_{12}, x_{21}, x_{22} exist so that for some θ ($\neq 0, \infty$),

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \theta\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Forming the product matrix, we obtain the equations

$$\begin{aligned} a_{11}x_{11} + a_{12}x_{21} &= \theta b_{11}, & a_{11}x_{12} + a_{12}x_{22} &= \theta b_{12}, \\ a_{21}x_{11} + a_{22}x_{21} &= \theta b_{21}, & a_{21}x_{12} + a_{22}x_{22} &= \theta b_{22}. \end{aligned}$$

These equations are satisfied by

$$\begin{aligned} x_{11} &= a_{22}b_{11} - a_{12}b_{21}, & x_{12} &= a_{22}b_{12} - a_{12}b_{22}, \\ x_{21} &= -a_{21}b_{11} + a_{11}b_{21}, & x_{22} &= -a_{21}b_{12} + a_{11}b_{22}. \end{aligned}$$

Hence X is the projectivity whose matrix is the product

$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

The first of these two matrices is proportional to the inverse of the matrix of A.

Similarly, there exists a projectivity Y, with matrix

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix},$$

such that $YA = B$.

(c) The identity element I of the group is the projectivity $\theta' : \phi' = \theta : \phi$ with matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The inverse of the projectivity A (as defined in group theory) is given by

$$\theta' : \phi' = a_{22}\theta - a_{12}\phi : -a_{21}\theta + a_{11}\phi$$

with matrix

$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

(d) The group of projectivities is not abelian. In fact the product AB has matrix

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

which, in general, cannot be reduced to the matrix of BA by multiplying each element by any common factor

Ex. 1 Taking projectivities to be defined by bilinear equations, show that

(i) the identity projectivity is given by $\lambda = \mu$;

• (ii) the inverse of the projectivity given by the equation $a\lambda\mu + b\lambda + c\mu + d = 0$ is given by $a\lambda\mu + b\mu + c\lambda + d = 0$, and is thus the same as the inverse defined in part (ii) of this section

Ex. 2. The inverse of the projectivity given by the equation

$$\mu = \frac{p\lambda + q}{r\lambda + s} \text{ is given by } \mu = \frac{-s\lambda + q}{r\lambda - p}.$$

Ex. 3 The equation of the projectivity T_i in which the points with parameters $0, \lambda_i, \infty$ correspond to the points with parameters $0, \mu_i, \infty$ (two points thus remaining unaltered) is $\lambda\mu_i = \lambda_i\mu$. Show that the inverse of T_i is $\lambda\lambda_i + \mu\mu_i = 0$ and that the projectivities T_i, T_j are commutative.

Ex. 4. The aggregate of projectivities on a line in which two given points correspond to themselves is a sub-group of the group of all projectivities on the line.

Ex. 5. The projectivity given by $a\lambda\mu + b\lambda + c\mu + d = 0$ is its own inverse if and only if $b = c$.

(iv) The united points of a projectivity.—Since we have to deal here with a quadratic equation, possibly having complex roots, we consider only a modified complex plane; but the reader will have no difficulty in deciding how much of what is said applies to a modified real plane.

A point which corresponds to itself in a projectivity on a given line is called a *united point* of the projectivity. If the projectivity is defined by the bilinear equation

$$a\lambda\mu + b\lambda + c\mu + d = 0,$$

the united points are given by the equation

$$a\lambda^2 + (b + c)\lambda + d = 0$$

In general, therefore, a projectivity has two distinct united points If $(b + c)^2 = 4ad$, there is only one united point and the projectivity is called *parabolic*; both because the quadratic in λ has a repeated root and because the parabolic case is a limit of the general case, we say that the united point *counts twice* for a parabolic projectivity or that there are *two coincident united points*

A particular case, as regards the algebra, arises when $a = 0$ The equation giving the united points is then to be taken as $0 . \lambda^2 + (b + c)\lambda + d = 0$, the roots of which are ∞ and $-d/(b + c)$. If also $b + c = 0$, the united points are given by $0 . \lambda^2 + 0 . \lambda + d = 0$, the point for which $\lambda = \infty$ counts twice as a united point This interpretation may be seen to follow at once if the projectivity is defined in the homogeneous form by $\theta' : \phi' = -(b\theta + d\phi) : (a\theta + c\phi)$.

The quadratic in λ which gives the united points is satisfied identically if and only if $a = b + c = d = 0$, that is if and only if the projectivity is identity. Hence, a projectivity with three or more united points is necessarily identity

Ex. 6. The equation of a projectivity having just two distinct united points with finite parameters α, β has the form

$$\frac{\lambda - \alpha}{\lambda - \beta} = k\frac{\mu - \alpha}{\mu - \beta}$$

with $k \neq 0$, 1 or ∞.

If one united point has finite parameter α and the other has parameter ∞, the equation has the form

$$\lambda - \mu + k(\mu - \alpha) = 0,$$

with $k \neq 0$, 1 or ∞.

Ex. 7. The equation of a parabolic projectivity whose united point has finite parameter α has the form

$$\frac{1}{\lambda - \alpha} = \frac{1}{\mu - \alpha} + \frac{1}{k}$$

with $k \neq 0, \infty$.

If the united point has parameter ∞, the equation has the form

$$\lambda = \mu + k$$

with $k \neq 0, \infty$.

Ex. 8. If U, V are distinct united points of a projectivity in which A, B correspond to A', B' respectively, there is another projectivity, also with U, V as united points, in which A, A' correspond to B, B' respectively.

Ex. 9. If U is the united point of a parabolic projectivity in which A, B correspond to A', B' respectively, there is a parabolic projectivity, also with united point at U, in which A, A' correspond to B, B' respectively.

Ex 10. Two projectivities with the same pair of distinct or coincident united points are commutative (*cf.* *Ex.* 4).

Ex. 11. A parabolic projectivity can have its equation expressed in the form $\lambda\mu + (\alpha + \beta)\lambda + (\alpha - \beta)\mu + \alpha^2 = 0$, the parameter of the united point being $-\alpha$.

(v) Periodic projectivities.

(*a*) Let n be a positive integer. The *nth power* of a projectivity T is defined inductively by the equalities

$$T^1 = T,\ T^n = TT^{n-1} \text{ for } n > 1.$$

Clearly T^n is also a projectivity

We prove that also $T^n = T^{n-1}T$ for $n > 1$. This is obvious when $n = 2$; and when $n = 3$ we have $T^3 = TT^2 = T(TT) = (TT)T = T^2T$. We proceed inductively and now assume that the statement is true when $n = m - 1$, then

$$T^{m+1} = TT^m = T(TT^{m-1}) = T(T^{m-1}T) = (TT^{m-1})T = T^mT.$$

Thus the truth of the statement when $n = m - 1$ implies its truth when $n = m + 1$; it is true when $n = 2, 3$ and is therefore true for all $n > 1$.

We define T^0 to be the identity projectivity I. Then $T^1 = TT^0 = T^0T$.

We define T^{-n} to be the projectivity inverse to T^n, so that $T^nT^{-n} = T^{-n}T^n = I$. We prove that $T^{-n} = (T^{-1})^n$. This is true when $n = 1$; we assume that it is true when $n = m - 1$. Since

$$I = T^{-m}T^m = T^{-m}(TT^{m-1}) = (T^{-m}T)T^{m-1},$$

we have $$T^{-m}T = T^{-(m-1)} = (T^{-1})^{m-1}.$$

Therefore

$$T^{-m} = T^{-m}I = T^{-m}(TT^{-1}) = (T^{-m}T)T^{-1} = (T^{-1})^{m-1}T^{-1} = (T^{-1})^m.$$

The statement therefore follows by induction.

It may now be left to the reader to prove that, if r, s are any integers, positive, negative or zero,

$$T^{r+s} = T^rT^s = T^sT^r,$$
$$T^{rs} = (T^r)^s = (T^s)^r.$$

(b) A projectivity T such that $T^n = I$, n being a non-zero integer, is said to be *periodic* If m is the least positive value of n for which $T^n = I$, we call m the *period* of T

Let $T^k = I$, we prove that m divides k We may express k in the form $qm + r$, where q, r are integers and $0 \leqslant r < m$. Then we have

$$I = T^k = T^{qm+r} = (T^m)^qT^r = I^qT^r = T^r.$$

By the definitions of m and of T^0 it follows that $r = 0$; that is m divides k

Thus every integral power of T is one or other of the projectivities $I = T^0, T, T^2, \ldots, T^{m-1}$ This set of projectivities is clearly a sub-group of the set of all the projectivities on the line It is called the *cyclic sub-group generated by* T, its order is m

Ex 12. A necessary and sufficient condition for the projectivity defined by $a\lambda\mu + b\lambda + c\mu + d = 0$ to be (i) of period 2 is $b = c$, (ii) of period 3 is $b^2 + c^2 = ad + bc$

Ex 13. A necessary and sufficient condition for the projectivity defined by $\theta' \;\; \phi' = a_{11}\theta + a_{12}\phi \;\; a_{21}\theta + a_{22}\phi$ to be (i) of period 2 is $a_{11} + a_{22} = 0$, (ii) of period 3 is $a_{11}{}^2 + a_{22}{}^2 + a_{11}a_{22} + a_{12}a_{21} = 0$

Ex. 14 Let ω be a complex pth root of 1, where p is a prime number Prove that the projectivity given by

$$\frac{\lambda - \alpha}{\lambda - \beta} = \omega\frac{\mu - \alpha}{\mu - \beta}$$

has period p

Ex 15 There is no periodic parabolic projectivity. [This is easily proved by means of the result of *Ex.* 7.]

(vi) Involutions.

(a) A projectivity T of period 2 is called an ***involutory projectivity*** or *involution.* It is self-inverse and is characterised by the property that if $T(P) = Q$ then $T(Q) = P$ for all points P. It is given by a bilinear equation of the form

$$a\lambda\mu + b(\lambda + \mu) + d = 0,$$

with $b^2 \neq ad$. Therefore it is determined by two pairs of corresponding points, which are now called *mates* in the involution. If the two pairs of mates have parameters λ_1, μ_1 and λ_2, μ_2 the equation of the involution which they determine is

$$\begin{vmatrix} \lambda\mu & \lambda + \mu & 1 \\ \lambda_1\mu_1 & \lambda_1 + \mu_1 & 1 \\ \lambda_2\mu_2 & \lambda_2 + \mu_2 & 1 \end{vmatrix} = 0$$

or
$$\frac{(\mu_1 - \lambda_2)(\lambda - \lambda_1)}{\lambda - \lambda_2} + \frac{(\lambda_1 - \mu_2)(\mu - \mu_1)}{\mu - \mu_2} = 0.$$

The united points of an involution are called the *double points* of the involution; since $b^2 \neq ad$ they are always distinct.

Ex 16. The equation of the involution whose double points have the finite parameters α, β is

$$\lambda\mu - \tfrac{1}{2}(\lambda + \mu)(\alpha + \beta) + \alpha\beta = 0.$$

If one double point has finite parameter α and the other has parameter ∞, the equation is

$$\lambda + \mu = 2\alpha$$

Ex. 17. A necessary and sufficient condition that a projectivity T should be an involution is that two distinct points, P, Q, exist such that $P = T(Q)$, $Q = T(P)$

Ex 18. The pairs of points P, P', Q, Q', R, R' are pairs of mates in an involution if and only if there is a projectivity in which P, P', Q, R correspond respectively to P', P, Q', R'

Ex. 19. An involution is determined by one double point and a pair of distinct mates or by the two double points.

(*b*) A useful property of involutions on the same line is that any two have just one pair of mates in common. If the equations of the involutions are

$$a\lambda\mu + b(\lambda + \mu) + d = 0, \; a'\lambda\mu + b'(\lambda + \mu) + d' = 0,$$

the common pair of mates is given by

$$\lambda\mu \,.\, \lambda + \mu : 1 = bd' - b'd : da' - d'a \cdot ab' - a'b$$

and the parameters of the two points are the roots of the quadratic in x

$$\begin{vmatrix} x^2 & -x & 1 \\ d & b & a \\ d' & b' & a' \end{vmatrix} = 0.$$

If the parameterisation is effected so that the double points of one involution have parameters 0, ∞ and those of the other

involution have parameters α, β then the common mates are easily shown to have the parameters $\pm(\alpha\beta)^{\frac{1}{2}}$.

(c) We prove that two involutions T, T' are commutative when the double points of either are mates in the other.

Let the equations of T, T' be respectively

$$a\lambda\mu + b(\lambda + \mu) + d = 0, \ a'\lambda\mu + b'(\lambda + \mu) + d' = 0.$$

The parameters of the double points of T are the roots of the quadratic $ax^2 + 2bx + d = 0$; therefore these points are mates in T' if and only if $ad' + a'd = 2bb'$; by the symmetry of this result, the double points of T' are then mates in T.

The equation of $T'T$ is easily found to be

$$\mu = -\frac{\lambda(a'd - bb') + (b'd - bd')}{\lambda(a'b - ab') + (bb' - ad')}.$$

Since $a'd - bb' = bb' - ad'$, the equation may be rewritten as

$$\mu = -\frac{\lambda(ad' - bb') + (bd' - b'd)}{\lambda(ab' - a'b) + (bb' - a'd)}$$

which represents TT'.

It is to be observed that TT' is also an involution.

The algebra is more expressive if we take the double points of T to have parameters $0, \infty$, so that T has the equation $\lambda + \mu = 0$ The hypothesis then implies that T' has an equation of the form $\lambda\mu = k$, $(k \neq 0, \infty)$. $TT' = T'T$ is the involution given by $\lambda\mu + k = 0$

(d) We prove that every non-involutory projectivity T may be expressed as the product T_1T_2 of two involutions.

Let P be any point which is not a united point of T and let $T^{-1}(P) = Q$, $T(P) = R$, then P, Q, R are distinct points. We take T_1 to be the involution having P as one double point and Q, R as a pair of mates Then

$$T_1T(Q) = T_1(P) = P, \quad T_1T(P) = T_1(R) = Q.$$

The projectivity T_1T is therefore an involution, say T_2. Therefore

$$T = IT = T_1^2T = T_1(T_1T) = T_1T_2.$$

If T is not parabolic we may take its united points U, V to have parameters $0, \infty$ respectively and take $P \equiv U + \theta V$. T has then an equation of the form $\lambda = k\mu$, $(k \neq 0, \infty)$ and T_1 is given by $\lambda\mu = \theta^2$. Then $T_2 = T_1T$ is given by $\lambda\mu = k\theta^2$.

If T is parabolic we may take its united point to have para-

meter ∞; then the equation of T has the form $\lambda = \mu + k$, $(k \neq 0, \infty)$. If P has parameter θ, the equations of T_1, T_2 are found to be respectively $\lambda + \mu = 2\theta$ and $\lambda + \mu = 2\theta + k$.

(e) We deal now with a particular case of a theorem due to Luroth which is investigated later in the general case.

The equation

$$(a\theta^2 + 2b\theta + c) + \phi(a'\theta^2 + 2b'\theta + c') = 0,$$

where $a : b . c \neq a' \ b' : c'$, determines two values of θ for every value of ϕ These values, say λ, μ, of θ are, in general, connected by an involutory bilinear equation We have

$$\begin{aligned}(a + \phi a')(\lambda + \mu) &= -2(b + \phi b'),\\ (a + \phi a')\lambda\mu &= c + \phi c';\end{aligned}$$

therefore, eliminating ϕ,

$$2(ab' - a'b)\lambda\mu + (ac' - a'c)(\lambda + \mu) + 2(bc' - b'c) = 0,$$

in which equation at least two coefficients are not zero. In order that the condition of non-singularity may hold, we require

$$(ac' - a'c)^2 - 4(ab' - a'b)(bc' - b'c) \neq 0$$

which is the case if and only if the quadratic in θ has no fixed root, independent of ϕ.

Conversely, let an involution be given with equation

$$A\lambda\mu + B(\lambda + \mu) + D = 0.$$

To every pair of mates with parameters λ, μ there corresponds a number ϕ such that

$$\begin{aligned}B(\lambda + \mu) &= -\tfrac{1}{2}D - \phi,\\ A\lambda\mu &= -\tfrac{1}{2}D + \phi\end{aligned}$$

Therefore λ, μ are the roots of the quadratic in θ

$$\theta^2 + \theta B^{-1}(\tfrac{1}{2}D + \phi) + A^{-1}(-\tfrac{1}{2}D + \phi) = 0,$$

that is

$$(2AB\theta^2 + AD\theta + BD) + 2\phi(A\theta + B) = 0$$

which is of the form stated at the beginning of this note.

We have thus reached an alternative form in which to express the equation of an involution. This new form clearly indicates the fact that the pairs of mates in the involution are in (1, 1) correspondence with the points of a line, the unordered pairs of numbers λ, μ being in (1, 1) correspondence with the numbers ϕ.

Ex. 20. The equation of an involution whose double points have parameters α, $-\alpha$ may be put in the form $(\theta-\alpha)^2+\phi(\theta+\alpha)^2=0$.

If the double points have parameters 0, ∞ the equation takes the form $\theta^2+\phi=0$, and if the double points have parameters α, β the equation takes the form $(\theta-\alpha)^2+\phi(\theta-\beta)^2=0$.

(vii) Projectivities between lines and pencils.

(*a*) In parts (ii), . . . , (vi) of this section we could equally well have taken the parameters λ, μ to be the parameters of lines in the pencil determined by two given lines (l_1, m_1, n_1), (l_2, m_2, n_2), every line of the pencil having co-ordinates of the form $(\theta l_1+l_2,\ \theta m_1+m_2,\ \theta n_1+n_2)$. In this dual way we define *projectivities*, with their *united lines*, and *involutions*, with their *double lines*, in a pencil.

The reader will benefit greatly by carrying out this process in the same detail as for a line.

More generally, by taking λ to be a parameter attached to a variable point on one line or to a variable line in one pencil and μ to be a parameter attached to a variable point on another line or to a variable line in another pencil, we may define *projectivities between two different lines* or *between two different pencils* or *between a line and a pencil.*

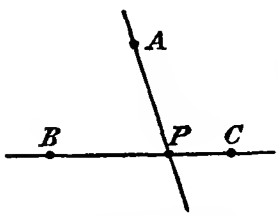


FIG 14—INCIDENCE PROJECTIVITY

(*b*) Let A, B, C be three distinct points which are not in line and let P be a variable point on BC (Fig 14) The correspondence between P and the line AP is clearly (1, 1); we prove that it is a projective correspondence Since corresponding elements are incident, we call this an *incidence projectivity*

We assign to A, B, C the triads of co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively. Then we may take a triad of co-ordinates for P to be $(\theta x_2+x_3,\ \theta y_2+y_3,\ \theta z_2+z_3)$, and a triad of co-ordinates for AP is consequently $(\theta l_2+l_3,\ \theta m_2+m_3,\ \theta n_2+n_3)$ where

$$\begin{aligned} l_2 &= y_1z_2-y_2z_1, & l_3 &= y_1z_3-y_3z_1,\\ m_2 &= z_1x_2-z_2x_1, & m_3 &= z_1x_3-z_3x_1,\\ n_2 &= x_1y_2-x_2y_1, & n_3 &= x_1y_3-x_3y_1. \end{aligned}$$

Thus P corresponds to AP in the projectivity whose equation is $\lambda=\mu$.

(*c*) We give a generalisation of incidence projectivity. Let $A_1, A_2, \ldots, A_n$ be n points and $\mathsf{L}_0, \mathsf{L}_1, \ldots, \mathsf{L}_n$ be $n+1$ lines,

such that A_i does not lie on L_{i-1} nor on L_i, for all i. $P_0, P_1, \ldots, P_n$ are $n+1$ variable points, one on each line taken in order, such that A_i, P_{i-1}, P_i are in line, for all i (Fig. 15). Then we assert that P_0 corresponds to P_n in a projectivity between $\mathsf{L}_0, \mathsf{L}_n$.

Formally the proof is inductive; it may be condensed as follows. Let $B_0, B_1, \ldots, B_n$ and $C_0, C_1, \ldots, C_n$ be two sets of positions of $P_0, P_1, \ldots, P_n$ respectively. Then, as in (b) above, we may assign co-ordinates in turn to B_0, C_0, to $A_1B_0 = A_1B_1$,

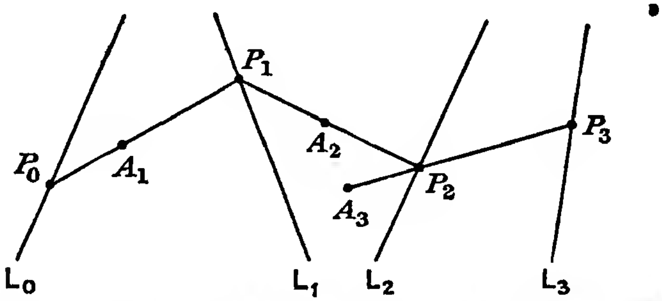


FIG 15.—GENERALISED INCIDENCE PROJECTIVITY IN THE CASE $n = 3$.

$A_1C_0 = A_1C_1$, to B_1, C_1, to $A_2B_1 = A_2B_2$, $A_2C_1 = A_2C_2, \ldots,$ to B_n, C_n so that if

$$P_0 \equiv \theta B_0 + C_0$$

then, in turn,

$$\begin{aligned}(A_1P_0) &\equiv \theta(A_1B_0) + (A_1C_0),\\ P_1 &\equiv \theta B_1 + C_1\\ (A_2P_1) &\equiv \theta(A_2B_1) + (A_2C_1),\\ &\cdots\cdots\cdots\\ P_n &\equiv \theta B_n + C_n.\end{aligned}$$

Thus P_0 corresponds to P_n in the projectivity between L_0 and L_n given by $P_0 \equiv \lambda B_0 + C_0$, $\lambda = \mu$, $P_n = \mu B_n + C_n$.

Ex. 21. State and prove the dual of the theorem in (c) above.

Ex. 22. If, in (c), we take $n = 2$ and L_2 to coincide with L_0, find the united points of the projectivity in which P_0 corresponds to P_2

(d) An important special case of the theorem in (c) is when $n = 1$. The projectivity between L_0 and L_1 is called a *perspective* and A_1 is called the *centre of perspective* (Fig 16).

Dually, the correspondence between two pencils of lines determined by corresponding lines intersecting on a given line A is

projective and is also called a perspective; the line A is called the *axis of perspective* (Fig. 17).

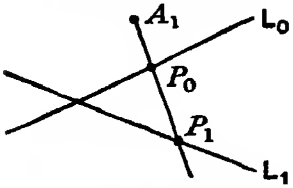


FIG. 16.—PERSPECTIVE BETWEEN LINES

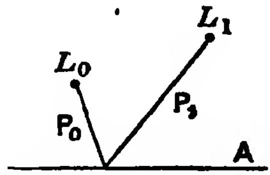


FIG. 17.—PERSPECTIVE BETWEEN PENCILS

Ex. 23. Let A, B, C, D be four points, no three being in line Let P be any point on AD, then let points Q, R, S, T, U, V be defined as follows:

$Q = BD.CP$, $R = CD.AQ$, $S = AD.BR$, $T = BD.CS$, $U = CD\ AT$, $V = AD\ BU$. Prove that V coincides with P by showing that the two points correspond in a projectivity having three united points

If we take $D \equiv A + B + C$ and $P \equiv \theta A + B + C$, show that we may take $Q \equiv \theta A + B + \theta C$, $R = A + B + \theta C$, $S \equiv A + \theta B + \theta C$, $T \equiv A + \theta B + C$, $U \equiv \theta A + \theta B + C$.

[This configuration, in the case where θ is a complex cube root of unity, is discussed again in section 55.]

10. Projective invariants: cross-ratio.

(i) Cross-ratio.

(*a*) An algebraic relation or geometrical entity which is unchanged by a particular projectivity is called an *invariant of the projectivity*; such, for example, are the united points. An invariant which does not depend on the particular projectivity of any group of projectivities is called an *invariant of the group*. If the group is the group of all projectivities between two sets of elements (for the present: points of a line or lines of a pencil), the invariant is called a *projective invariant*.

(*b*) Let λ_1, λ_2, λ_3, λ_4 be the parameters of four elements of one set and μ_1, μ_2, μ_3, μ_4 be the parameters of the corresponding elements, in any projectivity, of the other set.

Since the equation of the projectivity, regarded as determined by the first three of each set of four elements, is

$$\frac{(\lambda_1-\lambda_3)(\lambda-\lambda_2)}{(\lambda_3-\lambda_2)(\lambda_1-\lambda)} = \frac{(\mu_1-\mu_3)(\mu-\mu_2)}{(\mu_3-\mu_2)(\mu_1-\mu)}$$

we have

$$\frac{(\lambda_1-\lambda_3)(\lambda_4-\lambda_2)}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_4)} = \frac{(\mu_1-\mu_3)(\mu_4-\mu_2)}{(\mu_3-\mu_2)(\mu_1-\mu_4)}.$$

The number

$$\frac{(\lambda_1-\lambda_3)(\lambda_4-\lambda_2)}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_4)} \text{ or } \frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\Big/\frac{\lambda_1-\lambda_4}{\lambda_4-\lambda_3}$$

is therefore a projective invariant, its value being unchanged when we substitute $\mu_1, \mu_2, \mu_3, \mu_4$ for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ respectively. This number is denoted by $\{\lambda_1, \lambda_2;\ \lambda_3, \lambda_4\}$. The order of the numbers in this symbol is significant, λ_1, λ_2 are called an *associated pair*; so are λ_3, λ_4.

The number $\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\}$ is called both the *cross-ratio of the four numbers* $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, in the order written, and the *cross-ratio of the four elements*, in the same order, which these parameters represent. In order that the second definition should be significant, we must prove that the cross-ratio of the four elements is independent of the particular parameterisation set-up; this we now show.

We may suppose that the four elements are points P_1, P_2, P_3, P_4; the argument is similar in the dual case. Let, then, $P_i \equiv \lambda_i A + B$, $i = 1, 2, 3, 4$, with specified triads of co-ordinates for P_i, A, B, and let C, D be any two points on the line, also with specified co-ordinates.

There exist bonds $A \equiv \alpha_1 C + \alpha_2 D$, $B \equiv \beta_1 C + \beta_2 D$, with $\alpha_1 : \alpha_2 \neq \beta_1 : \beta_2$, and therefore

$$P_i \equiv (\lambda_i\alpha_1 + \beta_1)C + (\lambda_i\alpha_2 + \beta_2)D,\ i = 1, 2, 3, 4.$$

Thus P_i has also a triad of co-ordinates represented by $\lambda_i'C + D$, where

$$\lambda_i'(\lambda_i\alpha_2 + \beta_2) = \lambda_i\alpha_1 + \beta_1.$$

The non-singular bilinear equation

$$\lambda'(\lambda\alpha_2 + \beta_2) = \lambda\alpha_1 + \beta_1$$

therefore makes $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ correspond respectively to $\lambda_1', \lambda_2', \lambda_3', \lambda_4'$ and so $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{\lambda_1', \lambda_2', \lambda_3', \lambda_4'\}$.

The points C, D may be taken at A, B respectively, then it follows that the cross-ratio $\{P_1, P_2; P_3, P_4\}$ is not only independent of the positions of A, B but also of the triads of co-ordinates assigned to A, B. This cross-ratio is therefore dependent only on the points P_1, P_2, P_3, P_4 themselves.

(*c*) It should be observed here that any non-singular bilinear equation $p\lambda\lambda' + q\lambda + r\lambda' + s = 0$ gives rise to an alternative parameterisation on the line, λ' being the new parameter corresponding to the old parameter λ. Writing the equation in the form $\lambda'(\lambda p + r) = -q\lambda - s$, we see, by reference to the argument just given above, that this change of parameter is equivalent to a change of reference points on the line.

(*d*) Having remarked that the existence of a projectivity, in which the elements P_1, P_2, P_3, P_4 of one set correspond to the elements Q_1, Q_2, Q_3, Q_4 respectively in the other set, implies $\{P_1, P_2; P_3, P_4\} = \{Q_1, Q_2; Q_3, Q_4\}$, it is also important to observe that, conversely, if this equality holds between the two cross-ratios then such a projectivity necessarily exists.

The proof consists simply in that the equality of the cross-ratios is an algebraic statement of the fact that Q_4 is the element corresponding to P_4 in the projectivity which makes P_1, P_2, P_3 correspond to Q_1, Q_2, Q_3 respectively.

Ex 1. The bilinear equation in which λ_1, λ_2, λ_3 correspond to ∞, 0, 1 respectively makes λ_4 correspond to $\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\}$.

Ex. 2. Verify that

$$\{0, \infty; \lambda_3, \lambda_4\} = \lambda_3/\lambda_4, \quad \{\infty, 0; \lambda_3, \lambda_4\} = \lambda_4/\lambda_3.$$
$$\{\lambda_1, \lambda_2, 0, \infty\} = \lambda_1/\lambda_2, \quad \{\lambda_1, \lambda_2, \infty, 0\} = \lambda_2/\lambda_1.$$

Ex. 3. $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\{\lambda_1, \lambda_2, \lambda_4, \lambda_3\} = 1.$

Ex. 4 $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\{\lambda_1, \lambda_2, \lambda_4, \lambda_5\} = \{\lambda_1, \lambda_2; \lambda_3, \lambda_5\}.$

(ii) The six cross-ratios of four numbers.—Since four numbers can be permuted in twenty-four ways, it might be expected that twenty-four different cross-ratios could be derived from four numbers; in fact, only six cross-ratios may be derived The various cross-ratios may be classified as follows

First, there are six sets of four equal cross-ratios For example, it is easily verified that

$$\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\} = \{\lambda_3, \lambda_4; \lambda_1, \lambda_2\} = \{\lambda_2, \lambda_1; \lambda_4, \lambda_3\} = \{\lambda_4, \lambda_3; \lambda_2, \lambda_1\},$$

that is, a cross-ratio is unaltered when the two associated pairs are permuted, the orders in the pairs being preserved, and when the numbers are permuted simultaneously in the two associated pairs. Thus we need investigate only the six cross-ratios in which λ_1 stands first

Using the identity

$$(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4) + (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_4) + (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) \equiv 0,$$

we may verify without difficulty that, if $\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\} = k$, we have

$$\begin{array}{ll} \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = k, & \{\lambda_1, \lambda_2; \lambda_4, \lambda_3\} = 1/k, \\ \{\lambda_1, \lambda_3, \lambda_2, \lambda_4\} = 1 - k, & \{\lambda_1, \lambda_3, \lambda_4, \lambda_2\} = 1/(1 - k), \\ \{\lambda_1, \lambda_4, \lambda_2, \lambda_3\} = -(1 - k)/k, & \{\lambda_1, \lambda_4; \lambda_3, \lambda_2\} = -k/(1 - k). \end{array}$$

In general, these six cross-ratios are all different. Two are equal when $k = 0$; then $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_4$ or both equalities hold. Two others are equal when $k = \infty$; then $\lambda_1 = \lambda_4$ or

$\lambda_2 = \lambda_3$ or both As regards other values of k there are essentially two cases to consider.

If $k = 1/k$, we have $k = \pm 1$. With $k = 1$, we have $\lambda_1 = \lambda_2$ or $\lambda_3 = \lambda_4$ or both With $k = -1$, the four values of λ are distinct and we have

$$\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\} = \{\lambda_2, \lambda_1, \lambda_3, \lambda_4\} = \{\lambda_1, \lambda_2, \lambda_4, \lambda_3\} = \{\lambda_2, \lambda_1; \lambda_4, \lambda_3\}$$
$$= \{\lambda_3, \lambda_4; \lambda_1, \lambda_2\} = \{\lambda_4, \lambda_3; \lambda_1, \lambda_2\} = \{\lambda_3, \lambda_4, \lambda_2, \lambda_1\} = \{\lambda_4, \lambda_3; \lambda_2, \lambda_1\}.$$

If, then, $k = -1$, the cross-ratio $\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\}$ is unaltered by permuting the numbers in either of the associated pairs or by permuting the two pairs. We say that λ_1, λ_2 are *harmonic* with respect to λ_3, λ_4 and express this briefly by writing λ_1, λ_2 *harm* λ_3, λ_4. We say also that λ_4 is the *harmonic conjugate* of λ_3 with respect to λ_1, λ_2 and express this by writing $\lambda_4 = (\lambda_1, \lambda_2)/\lambda_3$. And we say that the cross-ratio itself is harmonic.

In virtue of the various permutations which leave a harmonic cross-ratio $\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\}$ unaltered, we have, of course, such relations as λ_3, λ_4 *harm* λ_1, λ_2 and $\lambda_3 = (\lambda_1, \lambda_2)/\lambda_4$.

The harmonic case arises naturally in connection with simple geometrical configurations and is of great importance.

There is one other case of equality amongst the six cross-ratios mentioned above, this occurs when $k = -(1 - k)/k$; that is, when $k = \omega$ or ω^2, where ω is a complex cube root of -1 Then, with either value for k,

$$\begin{aligned}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} &= \{\lambda_1, \lambda_4, \lambda_2, \lambda_3\} = \{\lambda_1, \lambda_3; \lambda_4, \lambda_2\}\\ &= \{\lambda_4, \lambda_2, \lambda_1, \lambda_3\} = \{\lambda_3, \lambda_2, \lambda_4, \lambda_1\}\\ &= \{\lambda_4, \lambda_1, \lambda_3, \lambda_2\} = \{\lambda_2, \lambda_4, \lambda_3, \lambda_1\}\\ &= \{\lambda_3, \lambda_1; \lambda_2, \lambda_4\} = \{\lambda_2, \lambda_3, \lambda_1, \lambda_4\},\end{aligned}$$

that is, the cross-ratio is unaltered by permuting cyclically any three of the four numbers, the fourth number being left unchanged Such a cross-ratio is called *equi-anharmonic* It has not the importance of the harmonic case since, if three of the numbers are real the fourth must be complex, thus the equi-anharmonic case does not occur in a real plane.

(iii) Harmonic cross-ratio.—The relation $\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\} = -1$ may be rearranged, if the four numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are all finite, in the form

$$(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) = 2(\lambda_1\lambda_2 + \lambda_3\lambda_4),$$

which exhibits clearly the association between λ_1 and λ_2 and between λ_3 and λ_4 and also the permissible permutations in the order of the four numbers in the symbol.

If $\lambda_1 = 0$ and $\lambda_2, \lambda_3, \lambda_4$ are finite and different from zero, the relation is $\{0, \lambda_2; \lambda_3, \lambda_4\} = -1$ which may be rearranged as

$$\frac{2}{\lambda_2} = \frac{1}{\lambda_3} + \frac{1}{\lambda_4},$$

showing that then λ_2 is the harmonic mean of λ_3, λ_4.

If $\lambda_1 = \infty$ and $\lambda_2, \lambda_3, \lambda_4$ are finite, the relation is $\{\infty, \lambda_2; \lambda_3, \lambda_4\} = -1$ which may be rearranged as

$$2\lambda_2 = \lambda_3 + \lambda_4,$$

showing that then λ_2 is the arithmetic mean of λ_3, λ_4. In particular, if also $\lambda_2 = 0$, we have $\lambda_3 = -\lambda_4$.

If $\lambda_1 = -\lambda_2$ and is neither 0 nor ∞, the relation is $\{\lambda_1, -\lambda_1; \lambda_3, \lambda_4\} = -1$ which may be expressed as

$$\lambda_3\lambda_4 = \lambda_1^2,$$

and then λ_1 is a geometric mean of λ_3, λ_4.

In all cases, if λ_1, λ_2 are the roots of $p\lambda^2 + 2q\lambda + r = 0$ and if λ_3, λ_4 are the roots of $p'\lambda^2 + 2q'\lambda + r' = 0$, a necessary and sufficient condition for $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = -1$ is $pr' + p'r = 2qq'$, this is easily proved.

Ex. 5. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be finite and correspond respectively to $\mu_1, \mu_2, \mu_3, \mu_4$ in the appropriate substitution below. Then if the substitutions are respectively

(i) $\mu = \lambda - \lambda_1$, (iii) $\mu = \lambda - \frac{1}{2}(\lambda_1 + \lambda_2)$,
(ii) $\mu = 1/(\lambda - \lambda_1)$, (iv) $\mu = (\lambda - \lambda_1)/(\lambda - \lambda_2)$,

the cross-ratio $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ becomes (i) $\{0, \mu_2; \mu_3, \mu_4\}$, (ii) $\{\infty, \mu_2, \mu_3, \mu_4\}$, (iii) $\{-k, +k,$ [illegible] $\mu_4\}$ where $k = \frac{1}{2}(\lambda_2 - \lambda_1)$, (iv) $\{0, \infty : \mu_3, \mu_4\}$.

Hence, or directly, show that, if λ_1, λ_2 *harm* λ_3, λ_4,

$$\text{(i), (ii)} \quad \frac{2}{\lambda_2 - \lambda_1} = \frac{1}{\lambda_3 - \lambda_1} + \frac{1}{\lambda_4 - \lambda_1},$$

$$\text{(iii)} \quad (2\lambda_3 - \lambda_1 - \lambda_2)(2\lambda_4 - \lambda_1 - \lambda_2) = (\lambda_1 - \lambda_2)^2,$$

$$\text{(iv)} \quad \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2} + \frac{\lambda_4 - \lambda_1}{\lambda_4 - \lambda_2} = 0.$$

(*b*) An important property of involutions is that the double elements are harmonic with respect to every pair of mates.

If the double elements have finite parameters α, β, the equation of the involution is

$$\lambda\mu - \tfrac{1}{2}(\lambda + \mu)(\alpha + \beta) + \alpha\beta = 0.$$

Rearranging this as

$$(\lambda + \mu)(\alpha + \beta) = 2(\lambda\mu + \alpha\beta),$$

we see, by what has been said in (*a*) above, that the mates with parameters λ, μ are harmonic with respect to the double elements; and the equation of the involution may be expressed concisely as $\{\lambda, \mu; \alpha, \beta\} = -1$.

If the double elements have parameters ∞, α, the equation of the involution is

$$\lambda + \mu = 2\alpha,$$

this, we have just seen, is equivalent to $\{\lambda, \mu; \infty, \alpha\} = -1$.

Ex. 6. U is the united point of a parabolic projectivity T; P being any other point, $T^{-1}(P) = P'$ and $T(P) = P''$. Prove that U, P *harm* P', P''

(iv) Metrical aspects of cross-ratio.

(*a*) All that has been said about cross-ratio applies to a modified real euclidean plane, subject only to the limitation to real numbers. We now discuss some metrical aspects of cross-ratio in such a plane.

Let A, B be two accessible points with co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and let P_i, $i = 1, 2, 3, 4$, be four points on the line AB, we start with co-ordinates for P_i given by $P_i \equiv \lambda_i A + B$

Since $z_1 z_2 \neq 0$, we choose a parameter θ_i for P_i given by $\theta_i z_2 = \lambda_i z_1$; then P_i has a triad of co-ordinates

$$\left(\theta_i \frac{x_1}{z_1} + \frac{x_2}{z_2},\ \ \theta_i \frac{y_1}{z_1} + \frac{y_2}{z_2},\ \ \theta_i + 1\right).$$

Referring to Joachimstal's formulae, we see that this is a triad of co-ordinates for the point which divides $\overrightarrow{AB}$ in the ratio $1 : \theta_i$, that is $\theta_i \overrightarrow{AP_i} = \overrightarrow{P_i B}$ or $\overrightarrow{AP_i}(1 + \theta_i) = \overrightarrow{AB}$, provided that P_i is accessible. If P_i is inaccessible, we have $\theta_i = -1$ and we agree to say that P_i *divides* $\overrightarrow{\mathrm{AB}}$ *externally in the ratio* $1 : -1$, though we do not go so far as to speak of a distance, sensed or unsensed, between P_i and any other point.

The equations connecting any two of λ_i, θ_i, $\overrightarrow{AP_i}$ are bilinear and non-singular; hence, for accessible points P_i,

$$\{P_1, P_2;\ P_3, P_4\} = \{\lambda_1, \lambda_2;\ \lambda_3, \lambda_4\} = \{\theta_1, \theta_2;\ \theta_3, \theta_4\} = \{\overrightarrow{AP_1}, \overrightarrow{AP};\ \overrightarrow{AP_3}, \overrightarrow{AP_4}\}.$$

Since the cross-ratio is independent of A, by taking A at P_1 we see that each of these expressions is equal to $\{0, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}, \overrightarrow{P_1P_4}\}$ which is the same as

$$\frac{\overrightarrow{P_1P_3}}{\overrightarrow{P_3P_2}} \Bigg/ \frac{\overrightarrow{P_1P_4}}{\overrightarrow{P_4P_2}}.$$

In particular, if $\{P_1, P_2;\ P_3, P_4\} = -1$, we have

$$\frac{2}{\overrightarrow{P_1P_2}} = \frac{1}{\overrightarrow{P_1P_3}} + \frac{1}{\overrightarrow{P_1P_4}},$$

and, if O is the mid point of P_1, P_2,

$$\overrightarrow{OP_3}\,.\,\overrightarrow{OP_4} = \overrightarrow{OP_1}^2 = \overrightarrow{OP_2}^2.$$

If one of the four points, say P_2, is inaccessible, we have

$$\begin{aligned}\{P_1, P_2;\ P_3, P_4\} &= \{\theta_1, -1,\ \theta_3, \theta_4\}\\ &= \frac{\theta_1-\theta_3}{\theta_3+1} \Big/ \frac{\theta_1-\theta_4}{\theta_4+1}\\ &= \left(\frac{1+\theta_1}{1+\theta_3} - 1\right) \Big/ \left(\frac{1+\theta_1}{1+\theta_4} - 1\right)\\ &= \left(\frac{\overrightarrow{AP_3}}{\overrightarrow{AP_1}} - 1\right) \Big/ \left(\frac{\overrightarrow{AP_4}}{\overrightarrow{AP_1}} - 1\right)\\ &= \overrightarrow{P_1P_3}/\overrightarrow{P_1P_4};\end{aligned}$$

and therefore, if $\{P_1, P_2;\ P_3, P_4\} = -1$, we have $\overrightarrow{P_1P_3} + \overrightarrow{P_1P_4} = 0$ or $\overrightarrow{P_3P_1} = \overrightarrow{P_1P_4}$, so that P_1 is the mid point of P_3, P_4.

(*b*) In view of the last result, we define the *mid point of two accessible points* A, B in a modified complex plane to be the point $O = (A, B)/U$, where U is the inaccessible point on the line AB If we assign co-ordinates $(x_1, y_1, 1)$, $(x_2, y_2, 1)$ to A, B respectively, then a triad of co-ordinates for O is $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), 1)$, with these triads $O \equiv \frac{1}{2}(A + B)$.

(v) Cross-ratio of four lines.

(*a*) We remarked above that the theory of cross-ratio applies also to sets of four lines in a pencil

The equations of four such lines $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3, \mathsf{L}_4$, in a modified real or complex plane, may be taken in the form $p + \lambda_i p' = 0$, $i = 1, 2, 3, 4$, where $p \equiv lx + my + nz = 0$ and $p' \equiv l'x + m'y + n'z = 0$ are the equations of two lines of the pencil. Then we have

$$\{\mathsf{L}_1, \mathsf{L}_2;\ \mathsf{L}_3, \mathsf{L}_4\} = \{\lambda_1, \lambda_2;\ \lambda_3, \lambda_4\}.$$

A useful case of this equality is that the lines given by $p = 0$, $p' = 0$ are harmonic with respect to the lines given by $p + \theta p' = 0$, $p - \theta p' = 0$; we have, in fact, $\{0, \infty;\ \theta, -\theta\} = -1$.

(*b*) Let four concurrent lines L_1, L_2, L_3, L_4 meet two transversals respectively in the sets of points P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4. From the incidence projectivities involved, we see at once that

$$\{P_1, P_2;\ P_3, P_4\} = \{\mathsf{L}_1, \mathsf{L}_2;\ \mathsf{L}_3, \mathsf{L}_4\} = \{Q_1, Q_2;\ Q_3, Q_4\}.$$

Dually, if four collinear points P_1, P_2, P_3, P_4 are incident respectively with two sets of four concurrent lines L_1, L_2, L_3, L_4 and M_1, M_2, M_3, M_4, then

$$\{\mathsf{L}_1, \mathsf{L}_2;\ \mathsf{L}_3, \mathsf{L}_4\} = \{P_1, P_2;\ P_3, P_4\} = \{\mathsf{M}_1, \mathsf{M}_2;\ \mathsf{M}_3, \mathsf{M}_4\}.$$

(*c*) In a modified real euclidean plane let L, L' be two lines meeting in an accessible point, and let B, B' be respectively the bisectors of the angles $\{\mathsf{L}, \mathsf{L}'\}$ and $\{\mathsf{L}', \mathsf{L}\}$ (Fig 18). We prove that L, L' *harm* B, B'.

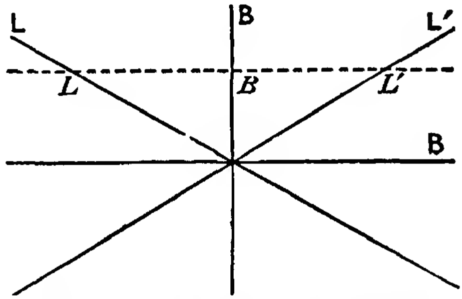


Fig 18.—The Bisectors of the Angles between Two Lines.

Let U be the inaccessible point on B'. Any other line through U meets L, L' in points L, L' and B in the mid point B of L, L' Therefore L, L' *harm* B, U; and hence, by incidence, L, L' *harm* B, B'.

Conversely, let L, L' and B, B' be four lines concurrent at an accessible point, then, if B, B' are perpendicular and L, L' *harm* B, B', the lines B, B' are the bisectors, in some order, of $\{\mathsf{L}, \mathsf{L}'\}$ and $\{\mathsf{L}', \mathsf{L}\}$. The proof amounts to a reversal of the argument just given.

(vi) Cross-ratio in relation to angles.

(*a*) We consider a real euclidean plane E and its covering modified complex plane E_M. In E we take a system of distance-co-ordinates x, y relative to a pair of rectangular axes; and in E_M we take the associated system of modified co-ordinates X, Y, Z.

Let L, L' be two intersecting lines in E. For simplicity, we may assume that the origin has been taken at their common point

and that the axes are different from both lines. The equations of L, L' are then of the form $y = mx$, $y = m'x$ respectively, with $0 \neq mm' \neq \infty$.

If θ denotes the angle $\{\mathsf{L}, \mathsf{L}'\}$, by section 1 (x) we have

$$\tan\theta = (m' - m)/(1 + mm').$$

The lines L, L' are covered by two lines L_M, L'_M respectively in E_M, these having equations $Y = mX$, $Y = m'X$. Through their common point pass two isotropic lines l, l' with equations $Y = iX$, $Y = -iX$. And we have

$$\begin{aligned}\{\mathsf{L}_M, \mathsf{L}'_M, \ \mathsf{l}, \mathsf{l}'\} &= \{m, m', \ i, -i\}\\ &= \frac{(m-i)(-i-m')}{(i-m')(m+i)}\\ &= \frac{(1+mm') - i(m'-m)}{(1+mm') + i(m'-m)}\\ &= \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\\ &= e^{-2i\theta} = e^{-2i(\theta + k\pi)},\end{aligned}$$

where k is any integer

Thus all the angles from L to L', in the sense of section 1 (x) are the roots of the equation in ϕ

$$e^{-2i\phi} = \{\mathsf{L}_M, \mathsf{L}'_M; \mathsf{l}, \mathsf{l}'\}.$$

Alternatively, we may write

$$\langle \mathsf{L}, \mathsf{L}'\rangle = \tfrac{1}{2}i\mathrm{Log}\,\{\mathsf{L}_M, \mathsf{L}'_M; \mathsf{l}, \mathsf{l}'\}.$$

Since $e^{-i\pi} = -1$, we have the important result that a necessary and sufficient condition that L, L' should be perpendicular is that L_M, L'_M *harm* l, l'.

(*b*) Now let L, L' denote any two lines in a modified complex euclidean plane which meet in an accessible point and let l, l' be the isotropic lines through their common point, it is to be specified, arbitrarily, at the outset, which circular point is to be contained by l and which by l'.

Every value of $\frac{1}{2}i$ Log $\{\mathsf{L}, \mathsf{L}'; \mathsf{l}, \mathsf{l}'\}$ is called an *interval from* L *to* L'; and any two values differ by an integral multiple of π. If L, L' cover real lines in the embedded real plane, it is clear that every interval from L to L' is also an angle from one real line to the other and *vice versa*; but the corresponding real lines will appear in the same order as L, L' only if l contains the appropriate circular point.

If M, M' are two other lines, intersecting in an accessible point, such that $\{\mathsf{L}, \mathsf{L}'; \mathsf{l}, \mathsf{l}'\} = \{\mathsf{M}, \mathsf{M}'; \mathsf{l}, \mathsf{l}'\}$, we say that the intervals from L to L' are equal to and have the *same sense* as the intervals

from M to M'. We say this because, if all four lines L, L', M, M' cover real lines, it then follows that the angles from the real line under L to the real line under L' have the same sense as the angles from the real line under M to the real line under M'. And, for a similar reason, if $\{\mathsf{L}, \mathsf{L}'; \mathsf{I}, \mathsf{I}'\} = \{\mathsf{M}, \mathsf{M}'; \mathsf{I}', \mathsf{I}\}$, we say that the intervals from L to L' have the *opposite sense* from the intervals from M to M'

On the basis of these ideas, we define two configurations in the complex plane to be *directly similar* (or *contra-similar*) if the elements of one figure may be put in (1, 1) correspondence with the elements of the other figure in such a way that corresponding intervals are equal and have the same (or opposite) sense. This definition agrees with the usual definition for a real euclidean plane.

In further developments of the theory, it is the cross-ratio $\{\mathsf{L}, \mathsf{L}', \mathsf{I}, \mathsf{I}'\}$ which is significant rather than the logarithm or interval

(c) The lines L, L' are said to be *perpendicular* if L, L' *harm* I, I'. Since the values of Log (-1) are the odd multiples of $\iota\pi$, all the intervals from L to L' (as also those from L' to L) are of the form $\frac{1}{2}\pi + k\pi$, where k is an integer. The definition thus agrees with the usual definition of perpendicularity for real lines in a real euclidean plane

If a system of co-ordinates is chosen as in (a) above, let L, L' have equations respectively

$$lX + mY + nZ = 0, \quad l'X + m'Y + n'Z = 0,$$

the coefficients being now real or complex. Then I is given by

$$(lX + mY + nZ) + \theta\,(l'X + m'Y + n'Z) = 0$$

where

$$(l + mj) + \theta(l' + m'j) = 0;$$

here j is one of the square roots of -1. The equation of I' is obtained by replacing j by $-j$. Hence

$$\begin{aligned} -1 &= \{\mathsf{L}, \mathsf{L}';\ \mathsf{I}, \mathsf{I}'\} \\ &= \{0, \infty;\ -(l + mj)/(l' + m'j),\ -(l - mj)/(l' - m'j)\} \\ &= \frac{(ll' + mm') - (lm' - l'm)j}{(ll' + mm') + (lm' - l'm)j}; \end{aligned}$$

and therefore $$ll' + mm' = 0.$$

Thus the relation $ll' + mm' = 0$, already familiar in connection with a real euclidean plane, continues to express a necessary and (obviously) sufficient condition for L, L' to be perpendicular. And we did, in fact, use this relation in section 2 as a definition for the perpendicularity of lines in a complex euclidean plane.

(*d*) Consider now any two non-parallel lines L, L′ and the two involutions in the pencil of lines containing L, L′, one having L, L′ as double lines and the other having I, I′ as double lines. These involutions have a common pair of mates B, B′, which are perpendicular since B, B′ *harm* I, I′.

If L, L′ cover real lines in the embedded real euclidean plane, it follows that B, B′ cover the bisectors of the angles between these real lines. For this reason we may here call B, B′ the *bisectors of the intervals* from L to L′ and from L′ to L.

If we parameterise every line in the pencil by giving it an equation of the form

$$(lX + my + nZ) + \lambda(l'X + m'Y + n'Z) = 0,$$

the equation of the involution having L, L′ as double lines is

$$\lambda + \mu = 0,$$

and the equation of the involution having I, I′ as double lines is

$$\lambda\mu + \tfrac{1}{2}(\lambda + \mu)\Big(\frac{l + mj}{l' + m'j} + \frac{l - mj}{l' - m'j}\Big) + \frac{l^2 + m^2}{l'^2 + m'^2} = 0.$$

Hence the parameters of B, B′ are given by

$$\lambda + \mu = 0, \quad \lambda\mu + \frac{l^2 + m^2}{l'^2 + m'^2} = 0.$$

and are therefore $\pm (l^2 + m^2)^{\frac{1}{2}}(l'^2 + m'^2)^{-\frac{1}{2}}$. Thus the equations of B, B′ are

$$\frac{lX + mY + nZ}{(l^2 + m^2)^{\frac{1}{2}}} \pm \frac{l'X + m'Y + n'Z}{(l'^2 + m'^2)^{\frac{1}{2}}} = 0.$$

11. Generalised homogeneous co-ordinates.

(i) The generalisation.

(*a*) We consider a modified euclidean plane which may be real or complex; numbers belong to the appropriate field of real or complex numbers

In a given system of modified co-ordinates, let (x, y, z) be a triad representing any point P. The equations

$$X : Y : Z = a_1x + b_1y + c_1z \,.\, a_2x + b_2y + c_2z \,.\, a_3x + b_3y + c_3z,$$

in which the coefficients are restricted so that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

determine a terset $[X, Y, Z]$, any of whose triads is called **a triad** of *generalised homogeneous co-ordinates* of P.

Since $\Delta \neq 0$, the equations are reversible, in the form

$$x : y : z$$
$$= A_1X + A_2Y + A_3Z : B_1X + B_2Y + B_3Z : C_1X + C_2Y + C_3Z,$$

where A_1 is the co-factor of a_1 in Δ, and so on, and

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \Delta^2 \neq 0.$$

The tersets $[x, y, z]$, $[X, Y, Z]$ are in $(1, 1)$ correspondence. Therefore any triad (X, Y, Z) of the terset $[X, Y, Z]$ serves equally as well as (x, y, z) to specify the point P. Since $\Delta \neq 0$, there is no point for which $X = Y = Z = 0$.

(*b*) In the new system of co-ordinates, a line is still characterised by a linear equation In fact, if the line is represented in the old system by

$$lx + my + nz = 0,$$

it is represented in the new system by

$$LX + MY + NZ = 0,$$

where

$$L : M : N$$
$$= A_1l + B_1m + C_1n : A_2l + B_2m + C_2n : A_3l + B_3m + C_3n.$$

We call (L, M, N) a triad of *generalised homogeneous co-ordinates* of the line

The equation of the inaccessible line in the new system is $C_1X + C_2Y + C_3Z = 0$

(*c*) Since the new co-ordinates are proportional to linear homogeneous polynomials in the old co-ordinates, it follows that every point P on the line joining the points A, B, with assigned new co-ordinates (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) respectively, has co-ordinates of the form $(\lambda X_1 + \mu X_2, \lambda Y_1 + \mu Y_2, \lambda Z_1 + \mu Z_2)$; and we can continue to express this fact by a bond $P \equiv \lambda A + \mu B$ A similar statement applies to lines.

(*d*) The lines $X = 0$, $Y = 0$, $Z = 0$ form a triangle, that they are not concurrent follows at once from the fact that $\Delta \neq 0$. This triangle is called the *fundamental triangle* or *triangle of reference* of the system of generalised co-ordinates. The point U for which $X : Y : Z = 1 : 1 : 1$ is called the *unit point* of the system; and the line U which $L : M : N = 1 : 1 : 1$ is called the *unit line*. The triangle ABC and unit point U together constitute a *frame of reference* for the system of generalised co-ordinates.

A system of generalised homogeneous co-ordinates is determined uniquely by prescribing a fundamental triangle and unit point (which, of course, must not lie on any side of the fundamental triangle). In fact, if in any of the old co-ordinate systems, the sides of the given triangle have the equations $a_ix + b_iy + c_iz = 0$, $i = 1, 2, 3$, and the unit point has a triad of co-ordinates (p, q, r), we simply put

$$X : Y : Z = \alpha_1(a_1x + b_1y + c_1z) : \alpha_2(a_2x + b_2y + c_2z) : \alpha_3(a_3x + b_3y + c_3z)$$

where $$\alpha_i = (a_ip + b_iq + c_ir)^{-1}.$$

(*e*) Let A, B, C be the vertices of the fundamental triangle respectively opposite to $X = 0$, $Y = 0$, $Z = 0$, these lines being named A, B, C. We may assign to A, B, C the co-ordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, then if a point P is assigned the triad (X, Y, Z), we have the bond $P \equiv XA + YB + ZC$. Similarly we may assign to A, B, C the same triads of line-co-ordinates, then if a line L is assigned the triad (L, M, N), we have the bond $\mathsf{L} \equiv L\mathsf{A} + M\mathsf{B} + N\mathsf{C}$.

(*f*) If in (*a*) above, the co-ordinates x, y, z are modified distance-co-ordinates, the plane now being supposed real, we may observe that X, Y, Z are proportional to certain multiples of the algebraic distances of P, in the case of an accessible point, from the lines A, B, C. If these algebraic distances are d_1, d_2, d_3 respectively, we have, for such a point,

$$X : Y : Z = d_1.\sqrt{a_1^2 + b_1^2} : d_2.\sqrt{a_2^2 + b_2^2} : d_3.\sqrt{a_3^2 + b_3^2}.$$

(ii) Cross-ratio in relation to generalised co-ordinates.—With the notation of part (i), let a particular point Q have a triad of co-ordinates (X_1, Y_1, Z_1) Then the four lines AB, AC, AU, AQ (Fig 19) have the equations $Z = 0$, $Y = 0$, $Y - Z = 0$, $YZ_1 - ZY_1 = 0$, respectively. Regarding these equations as being all of the form $Y + \lambda Z = 0$, we see that $\{AB, AC, AU, AQ\}$, which we abbreviate to $A\{B, C;\ U, Q\}$, is equal to $\{\infty, 0, -1, -Y_1/Z_1\}$, which is Y_1/Z_1. Thus, and in a similar way,

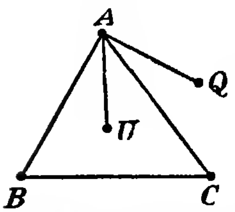


FIG. 19.—POINT IN RELATION TO FRAME OF REFERENCE.

$$A\{B, C;\ U, Q\} = Y_1/Z_1,$$
$$B\{C, A;\ U, Q\} = Z_1/X_1,$$
$$C\{A, B;\ U, Q\} = X_1/Y_1.$$

Similarly, if a line Q has a triad of co-ordinates (L_1, M_1, N_1), we have

$$\begin{aligned} \mathsf{A}\{\mathsf{B}, \mathsf{C};\ \mathsf{U}, \mathsf{Q}\} &= M_1/N_1, \\ \mathsf{B}\{\mathsf{C}, \mathsf{A};\ \mathsf{U}, \mathsf{Q}\} &= N_1/L_1, \\ \mathsf{C}\{\mathsf{A}, \mathsf{B};\ \mathsf{U}, \mathsf{Q}\} &= L_1/M_1, \end{aligned}$$

where, for example, A{B, C; U, Q} is short for {A . B, A . C; A . U, A . Q} and A . B stands for the point common to the lines A, B

(iii) Change of frame of reference.—Let (X, Y, Z) be generalised co-ordinates of a point P relative to the fundamental triangle ABC and unit point U; and let (X', Y', Z') be generalised co-ordinates of P relative to the fundamental triangle $A'B'C'$ and unit point U'

X, Y, Z are proportional to linear homogeneous polynomials in x, y, z (with the notation of part (i)), and x, y, z are proportional to linear homogeneous polynomials in X', Y', Z', each set of polynomials being a linearly independent set. Therefore X, Y, Z are proportional to linearly independent linear polynomials in X', Y', Z', and, similarly, the converse is true. Thus the two sets of co-ordinates are connected by equations of the form

$$X : Y : Z = \alpha_1 X' + \beta_1 Y' + \gamma_1 Z' : \alpha_2 X' + \beta_2 Y' + \gamma_2 Z' : \alpha_3 X' + \beta_3 Y' + \gamma_3 Z',$$

where
$$\Delta \equiv \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \neq 0,$$

which are reversed by

$$X' : Y' : Z' = A_1 X + A_2 Y + A_3 Z \cdot B_1 X + B_2 Y + B_3 Z : \Gamma_1 X + \Gamma_2 Y + \Gamma_3 Z,$$

where A_1 is the co-factor of α_1 in Δ, and so on.

Conversely, every such transformation corresponds to a change of frame of reference. Similar remarks also apply to line-co-ordinates. The proofs may be left to the reader.

(iv) Trilinear and areal co-ordinates.—We refer here to a modified real euclidean plane, and consider two particular systems of generalised homogeneous co-ordinates.

We have remarked that the co-ordinates X, Y, Z of an accessible point P are proportional to certain multiples of the algebraic distances of P from the sides of the triangle ABC. We now denote these distances by α_0, β_0, γ_0 respectively after first arranging that α_0 is positive for A, β_0 for B, and γ_0 for C (Fig. 20). Then, if Δ now denotes the area of the triangle ABC, a the unsensed distance $|BC|$, b $|CA|$, and c $|AB|$, we have

$$a\alpha_0 + b\beta_0 + c\gamma_0 = 2\Delta,$$

and $$X : Y : Z = u\alpha_0 : v\beta_0 : w\gamma_0,$$

where u, v, w are non-zero constants.

If we choose the unit point to be at the in-centre of the triangle ABC, we have there $\alpha_0 = \beta_0 = \gamma_0 = 1$ and therefore $u = v = w$. X, Y, Z are then called *trilinear co-ordinates* of P and are usually denoted by α, β, γ.

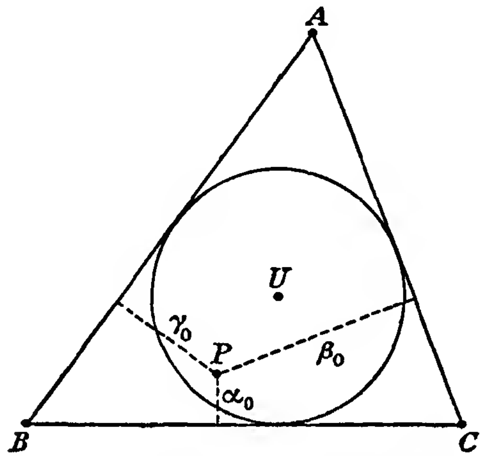


FIG 20.—TRILINEAR CO-ORDINATES

If we choose the unit point to be at the centroid of the triangle ABC, we have there $\alpha_0 : \beta_0 : \gamma_0 = a^{-1} : b^{-1} : c^{-1}$ and therefore $u : v : w = a : b : c$. X, Y, Z are then proportional to the areas (sensed in an obvious manner) of the triangles PBC, PCA, PAB, and are called *areal co-ordinates* (Fig 21).

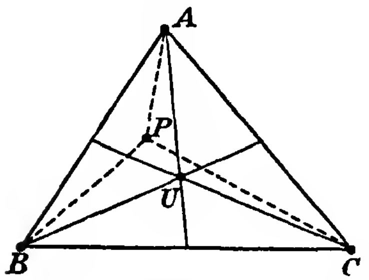


FIG. 21.—AREAL CO-ORDINATES.

Ex. 1. The locus of a point, the sum of whose algebraic distances from BC, CA, AB is constant, is the accessible part of a line. [The relation $\alpha_0 + \beta_0 + \gamma_0 = k$ is equivalent to $2\Delta(\alpha_0 + \beta_0 + \gamma_0) = k(a\alpha_0 + b\beta_0 + c\gamma_0)$, the equation of the line is therefore $2\Delta(\alpha + \beta + \gamma) = k(a\alpha + b\beta + c\gamma)$.]

Ex. 2. The trilinear equation of the inaccessible line is $a\alpha + b\beta + c\gamma = 0$. [Observe that the equation of the median through A is $b\beta = c\gamma$ and that, therefore, its harmonic conjugate

with respect to AB, AC has the equation $b\beta + c\gamma = 0$: this is the line through A parallel to BC. A line through the common inaccessible point on BC and this parallel is given by $a\alpha + b\beta + c\gamma = 0$; by symmetry this line passes through the inaccessible points on CA, AB; it is therefore the inaccessible line.]

Ex. 3. The areal equation of the inaccessible line is $X + Y + Z = 0$.

Ex. 4 A necessary and sufficient condition for the lines with trilinear equations $l\alpha + m\beta + n\gamma = 0$, $l'\alpha + m'\beta + n'\gamma = 0$ to be parallel is

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ a & b & c \end{vmatrix} = 0.$$

What is the corresponding condition relative to areal co-ordinates?

Ex 5. The algebraic distances of A, B, C from a line are respectively f, g, h. Prove that the trilinear equation of the line is $fa\alpha + gb\beta + hc\gamma = 0$.

Further remarks on trilinear co-ordinates will be found in section 37

12. Some linear configurations.

We are now ready to apply the algebraic preliminaries which have been set out in the preceding sections It is natural first to consider configurations of points and lines, these being associated with the algebra of linear homogeneous equations. What we have to say in this section applies equally to modified real and complex euclidean planes.

(i) Desargues' theorem for triangles in perspective.

(*a*) Two triangles ABC and $A'B'C'$ such that AA', BB', CC' are concurrent in a point O are said to be *in perspective* from O. Desargues' theorem is that the points $P = BC \,.\, BC'$, $Q = CA \,.\, C'A'$, $R = AB \;\, A'B'$ are in line (Fig 22).

We choose any system of homogeneous co-ordinates and refer to specific triads of co-ordinates for A, B, C, O Then we may take $A' \equiv A + aO$, $B' \equiv B + bO$, $C' \equiv C + cO$, for some a, b, c. P is on $B'C'$ and therefore is represented by a symbol of the form $\lambda(B + bO) + \mu(C + cO)$, P is also on BC and therefore this symbol depends only on B, C and not on O; hence $\lambda b + \mu c = 0$, and we may take a particular triad of co-ordinates for P so that $P \equiv b^{-1}B - c^{-1}C$. Similarly, we may take $Q = c^{-1}C - a^{-1}A$; $R \equiv a^{-1}A - b^{-1}B$. Thus $P + Q + R \equiv 0$, and so P, Q, R are in line

O is called the *centre of perspective* and PQR the *axis of perspective* of the two triangles.

Ex. 1. In the figure of Desargues' theorem, take ABC as triangle of reference and show that the equation of PQR is $ax + by + cz = 0$, using the notation of the text.

Ex. 2. Three triangles ABC, $A'B'C'$, $A''B''C''$ are in perspective from one point; taken in pairs they determine three axes of perspective; prove that these three axes are concurrent

(*b*) The converse of Desargues' theorem is as important as the direct theorem, of which it is in fact also the dual. The proof may therefore be adapted from the argument in (*a*) by using the symbols for lines just as we there used the symbols for points.

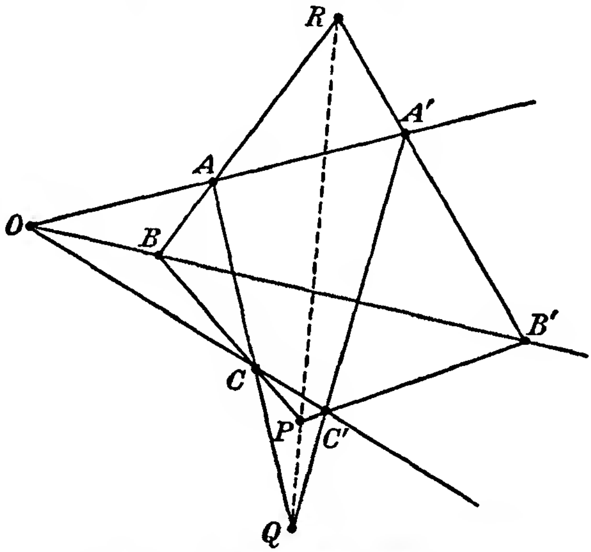


FIG. 22.—FIGURE FOR DESARGUES' THEOREM.

Another proof rests on the direct theorem, as follows. We start with the points $P = BC \,.\, B'C'$, $Q = CA \,.\, C'A'$, $R = AB \,.\, A'B'$ in line and observe that the triangles $BB'R$ and $CC'Q$ are in perspective from P. Therefore the points $A = BR \,.\, CQ$, $A' = B'R \,.\, C'Q$ are in line with the common point of BC, $B'C'$.

(ii) Pappus' theorem for triads of points on two lines.

(*a*) Let P, Q, R be three distinct points on one line and P', Q', R' be three distinct points on a different line, none of these six points being at the common point of the two lines. Pappus' theorem is that the points $F = QR' \,.\, Q'R$, $G = RP' \,.\, R'P$, $H = PQ' \,.\, P'Q$ are in line (Fig. 23).

Let the two given lines be AB, AC respectively and let us

refer to specific triads of co-ordinates for A, B, C in any system of homogeneous co-ordinates. Then we may take $P \equiv pA + B$, $P' \equiv p'A + C$, and so on.

F has symbols of both the forms $a(qA + B) + b(r'A + C)$ and $c(q'A + C) + d(rA + B)$. Therefore

$$\frac{aq + br'}{cq' + dr} = \frac{a}{d} = \frac{b}{c},$$

from which $c(q' - r') = d(q - r)$. Hence, we may take

$$F \equiv (qq' - rr')A + (q' - r')B + (q - r)C;$$

and, similarly,

$$\begin{aligned} G &\equiv (rr' - pp')A + (r' - p')B + (r - p)C, \\ H &\equiv (pp' - qq')A + (p' - q')B + (p - q)C. \end{aligned}$$

Therefore $F + G + H \equiv 0$; and F, G, H are in line.

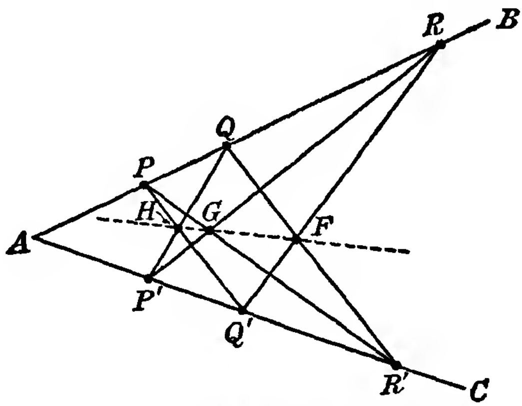


FIG 23.—FIGURE FOR PAPPUS' THEOREM.

(b) A particular case of interest is when the lines PP', QQ', RR' are concurrent in a point O. If then we choose B, C to be in line with O, we easily find that $p'/p = q'/q = r'/r = \theta$, say, and

$$\begin{aligned} (q - r)^{-1}F &\equiv \theta(q + r)A + (\theta B + C), \\ (r - p)^{-1}G &\equiv \theta(r + p)A + (\theta B + C), \\ (p - q)^{-1}H &\equiv \theta(p + q)A + (\theta B + C). \end{aligned}$$

In this case, then, F, G, H are all on the line joining A to the point $O' \equiv \theta B + C$. Further, it may be verified that $O \equiv -\theta B + C$; hence O, O' *harm* B, C. Thus, the axis of perspective is,

in this case, the harmonic conjugate of AO with respect to AB, AC (Fig. 24).

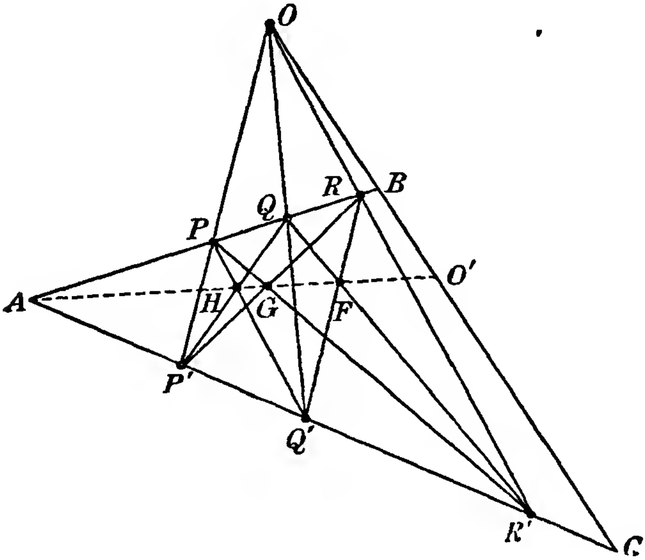


FIG 24.—SPECIAL CASE OF PAPPUS' THEOREM

(c) The dual of Pappus' theorem is as follows. Let P, Q, R be three distinct lines of one pencil and P′, Q′, R′ be three distinct lines of a different pencil, none of the six lines being the common line of the two pencils. Then if F joins Q . R′ and Q′ . R, G joins

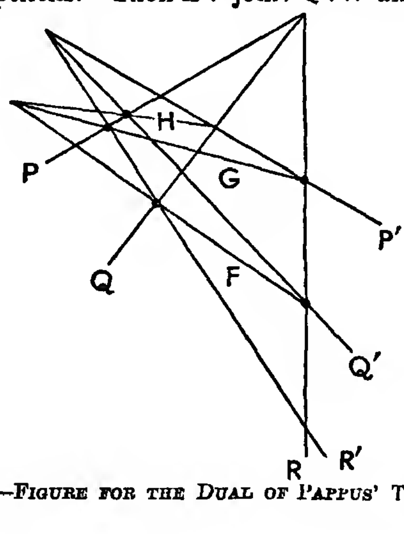


FIG. 25.—FIGURE FOR THE DUAL OF PAPPUS' THEOREM.

R . P′ and R′ . P, H joins P . Q′ and P′ . Q, the lines F, G, H are concurrent (Fig. 25).

Using the symbols of lines, the algebra of the proof is the same as in (*a*). The reader should carry out the proof and consider the particular case corresponding to (*b*).

Ex. 3. Obtain a theorem for a euclidean plane by considering the case of Pappus' theorem when R, R' are both inaccessible. Another theorem of interest arises from the dual of Pappus' theorem when the vertices of the two pencils are inaccessible.

(iii) Geometrical construction for a harmonic conjugate. The harmonic polar line of a point relative to a triangle.

(*a*) Let B, C, A' be three points in line; we give a geometrical construction for the point $(B, C)/A'$.

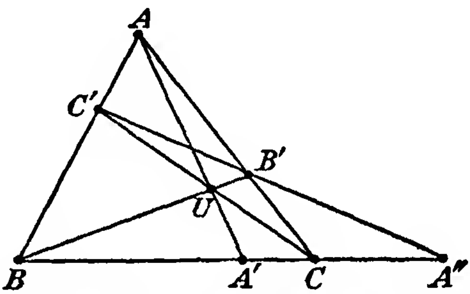


FIG 26 —CONSTRUCTION OF A HARMONIC CONJUGATE.

Let (Fig 26) A be any point, not on BC, and on AA' let U be any point, not at A or A'. Let BU meet CA at B', CU meet AB at C'. $B'C'$ meet BC at A''. Then A'' is the required point.

To prove this we remark that in any system of homogeneous coordinates there is a nul-bond of the form $aA + bB + cC + uU \equiv 0$ This may be expressed in the form $bB + cC \equiv -aA - uU$, each side in this bond is a symbol for one point which, being on both BC and AU is A'. We may therefore take $A' \equiv bB + cC$, and, similarly, $B' \equiv cC + aA$, $C' \equiv aA + bB$.

A'', being on $B'C'$, is represented by a symbol of the form $\lambda(cC + aA) + \mu(aA + bB)$; since A'' is also on BC, we have $\lambda + \mu = 0$. Therefore, we may take $A'' \equiv bB - cC$, which last is a symbol for the harmonic conjugate of the point with symbol $bB + cC$ with respect to B, C.

(*b*) Now (Fig. 27) let ABC be any triangle and U be any point, not on any side of the triangle. Let AU, BU, CU meet BC, CA, AB respectively in A', B', C' and let $A'' = (B, C)/A'$, $B'' = (C, A)/B'$, $C'' = (A, B)/C'$. We prove that A'', B'', C''

are in line; their line is called the ***harmonic polar line*** of U relative to the triangle ABC.

In any system of homogeneous co-ordinates there is a nul-bond of the form $aA + bB + cC + uU = 0$. Just as has been explained in (a) above, we may then take $A'' \equiv bB - cC$, $B'' \equiv cC - aA$, $C'' \equiv aA - bB$. Consequently $A'' + B'' + C'' \equiv 0$; and therefore A'', B'', C'' are in line.

Ex. 4. Taking ABC as fundamental triangle and U as the point (X, Y, Z), show that A', A'' have co-ordinates $(0, Y, Z)$, $(0, Y, -Z)$ respectively, and that the line $A''B''C''$ has co-ordinates (X^{-1}, Y^{-1}, Z^{-1}). In particular, if U is the unit point, $A''B''C''$ is the unit line and *vice versa.*

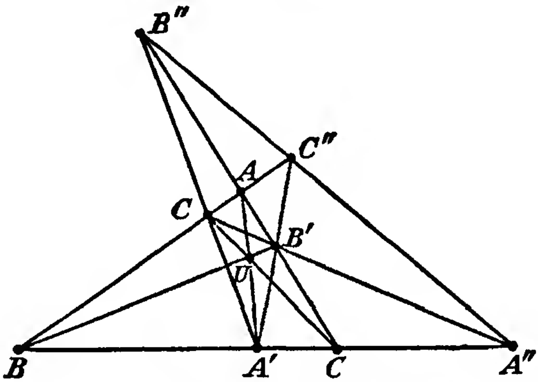


FIG. 27 —HARMONIC POLAR LINE.

(c) Before going further, it is convenient to introduce a common notation.

Let there be a projectivity between two sets of elements (which in either set may be all points on a line or all lines in a pencil) such that the elements $\alpha, \beta, \gamma, \ldots$ in one set correspond to the elements $\alpha', \beta', \gamma', \ldots$ respectively in the other set. We express this relationship by writing

$$(\alpha, \beta, \gamma, \ldots) \,\overline{\wedge}\, (\alpha', \beta', \gamma', \ldots).$$

In particular, if the elements are points on two lines and the projectivity is a perspective with centre A, we write $\overline{\wedge}_A$ to indicate the centre Similarly, if the elements are lines in two pencils and the projectivity is a perspective with axis A, we write $\overline{\wedge}_{\mathsf{A}}$ to indicate the axis.

The relation expressed by the symbol $\overline{\wedge}$ is reflexive, that is $(\alpha, \beta, \gamma, \ldots) \,\overline{\wedge}\, (\alpha', \beta', \gamma', \ldots)$ implies $(\alpha', \beta', \gamma', \ldots) \,\overline{\wedge}\, (\alpha, \beta, \gamma, \ldots)$; and it is transitive, that is if $(\alpha, \beta, \gamma, \ldots) \,\overline{\wedge}\, (\alpha', \beta', \gamma', \ldots)$ and

$(\alpha', \beta', \gamma', \ldots) \overline{\wedge} (\alpha'', \beta'', \gamma'', \ldots)$ then $(\alpha, \beta, \gamma, \ldots) \overline{\wedge} (\alpha'', \beta'', \gamma'', \ldots)$.

(*d*) The harmonic construction given in (*a*) may also be demonstrated as follows.

Let P be the point $AA' . B'C'$. Then

$$(B, C, A', A'') \overline{\wedge}_A (C', B', P, A'') \overline{\wedge}_U (C, B, A', A'').$$

Therefore B, C are mates in the involution having A', A'' as double points; hence B, C *harm* A', A''.

(*e*) It is important to observe that the geometrical construction for a harmonic conjugate point is independent of any system of co-ordinates. It therefore provides an alternative starting point for a theory of harmonic conjugates.

(*f*) Let B, C, A' be three concurrent lines. By dualising the construction in (*a*) we obtain the harmonic conjugate $\mathsf{A}'' = (\mathsf{B}, \mathsf{C})/\mathsf{A}'$ as follows.

We take any line A, not through $\mathsf{B} . \mathsf{C}$; and through $\mathsf{A} . \mathsf{A}'$ any line U, different from A, A'. Let B' join $\mathsf{B} . \mathsf{U}$ and $\mathsf{C} . \mathsf{A}$, C' join $\mathsf{C} . \mathsf{U}$ and $\mathsf{A} . \mathsf{B}$; then A'' joins $\mathsf{B}' . \mathsf{C}'$ and $\mathsf{B} . \mathsf{C}$.

Ex. 5. In the figure of (*a*) above, show that AA', AA'' harm AB, AC.

Ex 6. Using the harmonic conjugate construction, discuss the special case of Pappus' theorem (part (ii) (*b*) above).

Ex. 7. By dualisation of the last part of (*a*), define the *harmonic pole* of a line relative to a triangle and show that the line is the harmonic polar of its harmonic pole.

(iv) Quadrangles and quadrilaterals.

(*a*) Four distinct points A, B, C, D, no three of which are in line, are joined in pairs by six lines. The figure of points and lines is called a *quadrangle* and we refer to it as the quadrangle $ABCD$ (Fig. 28). Two lines such as AB, CD, together containing all four points, are called *opposite sides* of the quadrangle. The three points $E = AB . CD$, $F = AC . DB$, $G = AD . BC$, in which the pairs of opposite sides intersect, are called *diagonal points*; they are the vertices of the *diagonal triangle*

Dually, four distinct lines A, B, C, D, no three of which are concurrent, intersect in pairs in six points. The figure of lines and points is called a *quadrilateral* (Fig. 29). Two points such as $\mathsf{A} . \mathsf{B}$ and $\mathsf{C} . \mathsf{D}$ are called *opposite vertices*. The three lines which join the pairs of opposite vertices are called *diagonals*, they are the sides of the *diagonal triangle*.

We prove a number of properties of quadrangles and quadrilaterals, these give rise by duality to corresponding properties of quadrilaterals and quadrangles respectively.

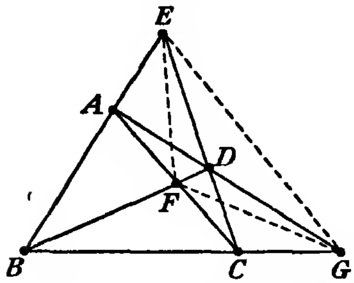


FIG 28 —QUADRANGLE WITH ITS DIAGONAL POINTS

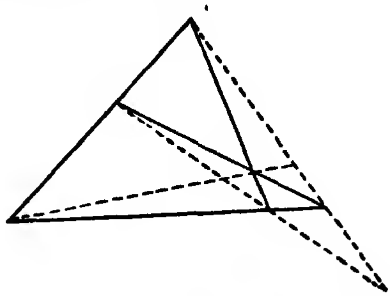


FIG 29.—QUADRILATERAL WITH ITS DIAGONALS

(*b*) The construction for a harmonic conjugate shows at once that any two opposite sides of the quadrangle $ABCD$ are harmonic with two sides of the diagonal triangle EFG.

Let FG meet AB, CD in P, Q, GE meet AC, DB in R, S; EF meet AD, BC in T, U (Fig. 30). Then Q, S, U are collinear,

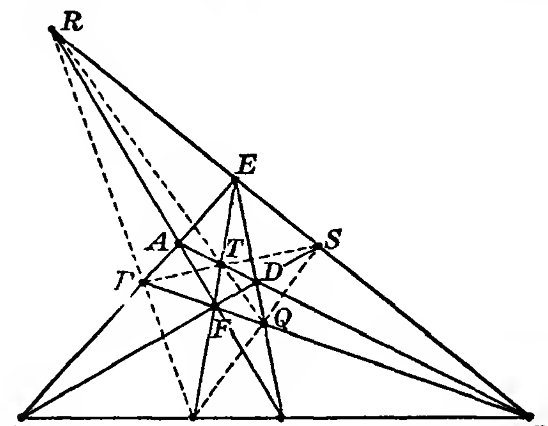


FIG 30 —FIGURE TO ILLUSTRATE HARMONIC PROPERTIES OF A QUADRANGLE.

being on the harmonic polar line A of A relative to the triangle BCD; Q, R, T are collinear, being on the harmonic polar line B of B relative to the triangle CDA; P, S, T are collinear, being on the harmonic polar line C of C relative to the triangle DAB; and P, R, U are collinear, being on the harmonic polar line D of D relative to the triangle ADC.

The four lines A, B, C, D in fact determine a quadrilateral in which P, Q; R, S; T, U are the pairs of opposite vertices; the diagonals of this quadrilateral are FG, GE, EF.

(*c*) Let us choose any system of homogeneous co-ordinates, taking specific triads of co-ordinates for A, E, F, G; then there is a bond $A \equiv xE + yF + zG$. We prove without difficulty that we may take

$$\begin{aligned} B &\equiv -xE + yF + zG, \\ C &\equiv +xE - yF + zG, \\ D &\equiv +xE + yF - zG; \end{aligned}$$

$$\begin{aligned} P &\equiv yF + zG, \quad & R &\equiv xE + zG, \quad & T &\equiv xE + yF, \\ Q &\equiv yF - zG, \quad & S &\equiv -xE + zG, \quad & U &\equiv xE - yF. \end{aligned}$$

The collinearity of, say, Q, S, U then follows from the nul-bond $Q + S + U \equiv 0$. Similarly we have $Q + R - T \equiv 0$, $-P + S + T \equiv 0$, $P - R + U \equiv 0$.

Ex. 8. Taking EFG as fundamental triangle, let A have a triad of co-ordinates (x, y, z). From the above bonds, write down co-ordinates for the other points in the figure. Show also that the lines A, B, C, D have co-ordinates (x^{-1}, y^{-1}, z^{-1}), $(-x^{-1}, y^{-1}, z^{-1})$, $(x^{-1}, -y^{-1}, z^{-1})$, $(x^{-1}, y^{-1}, -z^{-1})$ respectively.

Show further that, if a point O has co-ordinates (x_0, y_0, z_0), then the harmonic conjugate of EO with respect to the pair of opposite sides AB, CD has co-ordinates $(0, y_0 y^{-2}, -z_0 z^{-2})$ and that this and the two similar harmonic conjugate lines through F, G are concurrent in the point $(x^2 x_0^{-1}, y^2 y_0^{-1}, z^2 z_0^{-1})$.

State the dual form of the last result and deduce, as a particular case, that the mid points of the diagonals of a quadrilateral are in line. (The mid point of a diagonal is the mid point of the pair of opposite vertices on the diagonal).

(*d*) Referring again to the quadrangle $ABCD$, let a line L meet AB, CD in L, L'; AC, DB in M, M'; AD, BC in N, N' (Fig 31). We prove that L, L'; M, M', N, N' are pairs of mates in an involution on L.

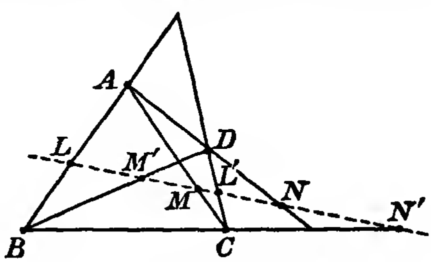


FIG 31.—INVOLUTION PROPERTY OF A QUADRANGLE.

Let BM meet CD at H and DM meet AB at K. Then $(M, M', L, N) \,\overline{\wedge}_D\, (K, B, L, A) \,\overline{\wedge}_M\, (D, H, L', C) \,\overline{\wedge}_B\, (M', M, L', N')$. Hence $(M, M', L, N) \,\overline{\wedge}\, (M', M, L', N')$ and therefore the statement follows.

Alternatively, let V, W be the double points of the involution determined by the pairs of mates L, L' and M, M'. We may take $L \equiv V + lW$, $L' \equiv V - lW$, $M \equiv V + mW$, $M' \equiv V - mW$, $N \equiv V + nW$, $B \equiv aA + V + lW$. Then it may be proved in turn that we may take $D \equiv a(m + n)A + (l + m)(V + nW)$, $C \equiv a(m + n)A + (n + l)(V + mW)$, $N' \equiv V - nW$ The form of the last result shows that N' is the mate of N in the involution.

Alternatively again, we may specify points on L by means of symbols $(x_1E + y_1F + z_1G) + \lambda(x_2E + y_2F + z_2G)$ where λ is a parameter. Then we may prove that, if L, L' and M, M' are pairs of mates defining the involution $a\lambda\mu + b(\lambda + \mu) + d = 0$, we have $a(y_1^2 - z_1^2) - 2b(y_1y_2 - z_1z_2) + d(y_2^2 - z_2^2) = 0$ and $a(z_1^2 - x_1^2) - 2b(z_1z_2 - x_1x_2) + d(z_2^2 - x_2^2) = 0$. From this it follows that $a(x_1^2 - y_1^2) - 2b(x_1x_2 - y_1y_2) + d(x_2^2 - y_2^2) = 0$ and hence that N, N' are mates in this involution.

Ex. 9. The pairs of lines which join the pairs of opposite vertices of a quadrilateral to a given point are pairs of mates in an involution.

Ex. 10. Referring to the text above, show that the theorem holds when L passes through a diagonal point, say E, then $L = L' = E$, which is a double point of the involution. If we take $M \equiv N + pE$, $M' \equiv N + qE$, prove that then $N + (p + q)E$ represents N'.

Ex. 11. In the general case of the text, we may assign coordinates to L, L', N so that $N \equiv L + L'$. If then we take $M \equiv L + pL'$, $M' \equiv L + qL'$, prove that $L + pqL'$ represents N'.

13. Geometrical constructions for projectivities and involutions.

(i) Projectivity on a line.—Let L be a line on which a projectivity T, which may be parabolic, has a united point U. We regard T as determined by U and two pairs of corresponding points A, $B = T(A)$ and A', $B' = T(A')$. We show how to construct geometrically the point $B'' = T(A'')$ corresponding to any given point A'' on L.

Through U we take a different line M (Fig. 32) and on this two different points P, P'. Let $H_1 = AP \,.\, A'P'$, $H_2 = BP \,.\, B'P'$, $P'' = H_1A'' \,.\, \mathsf{M}$; then $B'' = H_2P'' \,.\, \mathsf{L}$

We have $(U, A, A', A'') \overline{\wedge}_{H_1} (U, P, P', P'') \overline{\wedge}_{H_2} (U, B, B', B'')$. Hence $(U, A, A', A'') \overline{\wedge} (U, B, B', B'')$. Since a projectivity is determined uniquely by a united point and two pairs of corresponding points, it follows that T is the projectivity referred to in the last symbol $\overline{\wedge}$.

The second united point of T is the intersection of H_1H_2 with L. If, then, T is parabolic, H_1H_2 necessarily passes through U, and *vice versa*.

The construction depends on knowing a united point U of T. A construction which applies when T is regarded as determined by three pairs of corresponding points is given at the end of part (iii) of this section.

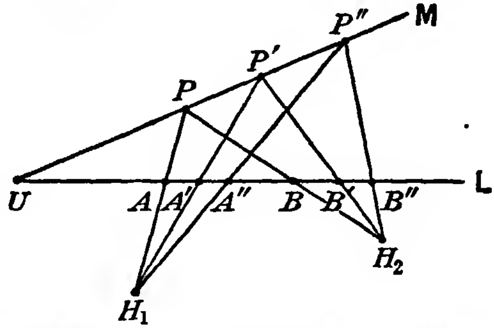


FIG 32.—GEOMETRICAL CONSTRUCTION FOR A PROJECTIVITY.

Ex. 1. Give the dual construction for a projectivity in a pencil

(ii) Involution on a line.—If in part (i) above, the projectivity T is involutory, we can suppose that the involution is determined by a double point U and a pair of mates A, B. Identifying A' with B and B' with A, the construction just given leads to two points H_1, H_2 which, with U, are the diagonal points of the quadrangle $ABPP'$ (Fig. 33). Let $A''H_1$ meet M in P''; then

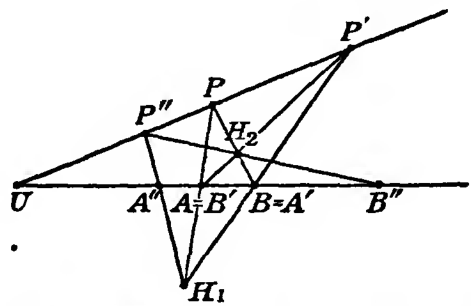


FIG. 33.—GEOMETRICAL CONSTRUCTION FOR AN INVOLUTION.

by the same kind of argument as before, $B'' = T(A'')$ is the point where H_2P'' meets L.

Since T is involutory, it follows that H_1B'', H_2A'' meet M in the same point Q; and U, H_1, H_2 are the diagonal points of the quadrangle $A''B''P''Q$. Let H_1H_2 meet L in V, then U, V harm A'', B'', and therefore V is the second double point of the involution.

Ex. 2. Give the dual construction for an involution in a pencil.

Ex. 3. Using the fact that a line is met by the pairs of opposite sides of a quadrangle in three pairs of mates of an involution, show how to construct the mate of a given point when two pairs of mates are given.

(iii) Projectivity between two different lines: the cross-axis.

(*a*) Let L, M be two given lines between which a projectivity T is determined by three points A, B, C on L corresponding to three points A', B', C' respectively on M.

By the theorem of Pappus, the three points $X = BC' . B'C$, $Y = CA' . C'A$, $Z = AB' . A'B$ lie on a line N (Fig. 34). Let P

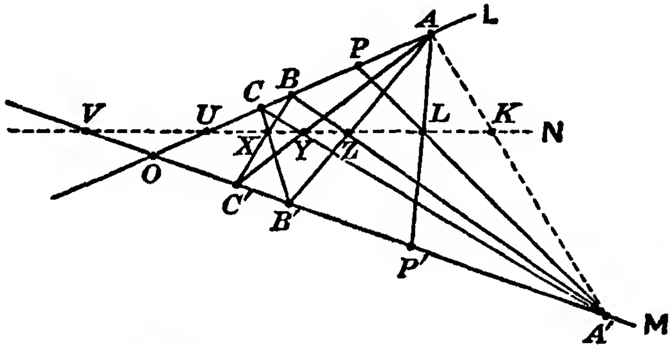


FIG. 34.—CROSS-AXIS OF A PROJECTIVITY BETWEEN TWO LINES.

be any point on L; let $A'P$ meet N in L and AL meet M in P'; we prove that $P' = T(P)$.

Let AA' meet N at K. Then $(A, B, C, P) \overline{\wedge}_{A'} (K, Z, Y, L) \overline{\wedge}_{A} (A', B', C', P')$. Therefore P' corresponds to P in the projectivity in which A', B', C' correspond respectively to A, B, C, that is in T.

The line N is called the *cross-axis* of T.

Ex 4. If P, Q are any two points on L and P', Q' are the corresponding points on M, then PQ' meets $P'Q$ on N.

(*b*) There are two cases to consider, according as N does or does not pass through O, the common point of L, M.

Let us suppose first that N passes through O (Fig. 35); and let AA' meet BB' at H, HP meet M in P''. From the quadrangle $AA'BB'$, N is the harmonic conjugate of OH with respect to L, M. Therefore, from the quadrangle $AA'P''P$, AP'' meets $A'P$

on N. Therefore AP'' is AP' and so P'' is P'. The projectivity in this case is therefore the perspective with centre at H. We note that O, regarded as a point of either L or M, corresponds to itself.

Let us now suppose that N does not pass through O but meets L at U and M at V. The construction just given shows that $O = T(U)$ and $V = T(O)$. We prove later that, in this (the more general) case, the lines PP' touch an irreducible conic which touches L at U and M at V.

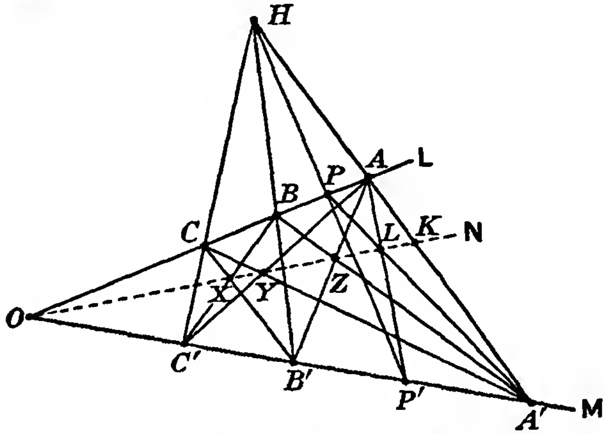


FIG 35—CROSS-AXIS OF A PERSPECTIVE BETWEEN TWO LINES.

(c) Consider now a projectivity T on a line L, T being determined by two triads of corresponding points A, B, C and A', B', C'. Let H be any point not on L and let HA', HB', HC' meet a line M at A'', B'', C'' respectively. Let T_1 be the projectivity which transforms A, B, C into A'', B'', C'' and T_2 the perspective which transforms A'', B'', C'' into A', B', C'.

Given any point P on L we construct, as in (a) above, the point $P'' = T_1(P)$; then we construct the point $P' = T_2(P'')$. It is simple to prove that $P' = T(P)$.

Ex. 5 A projectivity between two lines, in which the common point of the lines is self-corresponding, is necessarily a perspective

Ex. 6. Referring to the text in (a), if O is self-corresponding, the projectivity is expressed by an equation of the form $\lambda = k\lambda'$ when we take $P \equiv \lambda O + A$, $P' \equiv \lambda' O + A'$. Prove that we may take $H \equiv A - kA'$.

If O is not self-corresponding, let us take $P \equiv \lambda O + U$, $P' \equiv O + \lambda' V$, $A \equiv aO + U$, $A' \equiv O + a'V$. Then the projectivity is represented by the equation $\lambda a' = \lambda' a$. AP' meets UV at the point represented by $U - a\lambda' V$, and $A'P$ meets UV

at the point represented by $U - a'\lambda V$; these points are the same by reason of the equation of the correspondence.

Ex. 7. Give the dual construction for a projectivity between two pencils of lines and show that a necessary and sufficient condition for the line joining the vertices of the two pencils to be self-corresponding is that the projectivity should be a perspective. We prove later that the locus of the point common to corresponding lines is, in the general case, a conic passing through the vertices of the two pencils and touching there the lines which correspond in each pencil to the line joining the vertices.

(iv) Projectivity between a line and a pencil.—The problem of constructing a line corresponding to a given point, and *vice versa*, is now straightforward. In fact, it is sufficient to draw any transversal of the pencil and to consider the projectivity which arises between the given line and this new line. The construction of part (iii) above is involved.

14. Review of Chapter II.

The chapter has been concerned essentially with the geometrical interpretation of the algebraic theory of linear substitutions of the form $\lambda = (a\lambda' + b)/(c\lambda' + d)$ or, homogeneously and more symmetrically, of the form $\theta' : \phi' = p\theta + q\phi \quad r\theta + s\phi$ The resulting theory is of one-dimensional projective transformations.

Of special interest is the study of what is invariant with respect to projective transformations. This study is called *projective geometry* and is given prominence in this book *Metrical geometry*, in which attention is drawn to relations of length and angle, finds its place but is subordinated to projective geometry and is investigated chiefly with reference to the special parts played by the inaccessible circular points.

A numerical projective invariant of great importance is cross-ratio. We have seen that the special case of harmonic cross-ratio arises naturally in connection with involutory projectivities and with certain simple and fundamental geometrical configurations.

The theory of bonds has proved to be useful in solving algebraically, but without direct reference to any particular co-ordinate system, problems relating to the incidence of points and lines in various configurations. A loose definition of a bond, which may be helpful, is, for example, that the bond $P \equiv \lambda A + \mu B$ indicates that a triad of co-ordinates of P is obtainable by adding λ times a triad for A to μ times a triad for B.

Each of the matters considered is capable of wide generalisation, at the same time restricting the algebraic processes involved

to the use of homogeneous linear equations. We prefer at this stage to pass from linear equations to quadratic equations, as being the next simplest type, thus initiating the theory of conics. It will be shown that the co-ordinates of a variable point on a conic can be expressed as quadratic functions of a parameter; and much interesting geometry results from consideration of projective transformations of this parameter.

CHAPTER III

PROJECTIVE THEORY OF CONICS

15. Conics : preliminary properties.

(i) Definition.—The set of points in a modified complex euclidean plane whose co-ordinates, in any homogeneous system, satisfy an equation of the form

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

where the coefficients $a, \ldots, h$ belong to the field of complex numbers and are not all zero, is called a *conic*.

Since a change of frame of reference is represented by a linear transformation in the co-ordinates (section 11 (iii)), a conic is represented by the vanishing of a homogeneous quadratic polynomial in the co-ordinates in every system of homogeneous co-ordinates.

The conic is said to be *irreducible* or *proper* if S does not factorise; and to be *reducible* or *degenerate* if S does factorise, in which case the conic consists of two different lines or of a single line counted twice (section 5 (iii)). Some writers abbreviate the polynomial S to $(abcfgh)(xyz)^2$.

(ii) The fundamental quadratic.—Every point on the line joining two distinct points (x_1, y_1, z_1) and (x_2, y_2, z_2) has co-ordinates of the form $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2)$ Those points of the line which lie also on the conic have parameters determined by the quadratic in λ

$$\begin{aligned} &a(x_1 + \lambda x_2)^2 + 2f(y_1 + \lambda y_2)(z_1 + \lambda z_2) \\ &+ b(y_1 + \lambda y_2)^2 + 2g(z_1 + \lambda z_2)(x_1 + \lambda x_2) \\ &+ c(z_1 + \lambda z_2)^2 + 2h(x_1 + \lambda x_2)(y_1 + \lambda y_2) = 0, \end{aligned}$$

which we call the *fundamental quadratic* and write in the form

$$S_{11} + 2\lambda S_{12} + \lambda^2 S_{22} = 0,$$

where

$$\begin{aligned} S_{11} &\equiv ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 \\ &\equiv x_1(ax_1 + hy_1 + gz_1) + y_1(hx_1 + by_1 + fz_1) \\ &\qquad\qquad + z_1(gx_1 + fy_1 + cz_1) \\ &\equiv \tfrac{1}{2}\left(x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + z_1 \frac{\partial}{\partial z_1}\right) S_{11}; \end{aligned}$$

$$S_{12} \equiv ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1) + g(z_1x_2 + z_2x_1) + h(x_1y_2 + x_2y_1)$$
$$\equiv x_1(ax_2 + hy_2 + gz_2) + y_1(hx_2 + by_2 + fz_2) + z_1(gx_2 + fy_2 + cz_2)$$
$$\equiv x_2(ax_1 + hy_1 + gz_1) + y_2(hx_1 + by_1 + fz_1) + z_2(gx_1 + fy_1 + cz_1)$$
$$\equiv \tfrac{1}{2}\left(x_2\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1} + z_2\frac{\partial}{\partial z_1}\right)S_{11};$$
$$S_{22} \equiv ax_2^2 + by_2^2 + cz_2^2 + 2fy_2z_2 + 2gz_2x_2 + 2hx_2y_2.$$

In general, therefore, every line meets the conic in two distinct points (Fig. 36). If the fundamental quadratic has equal roots,

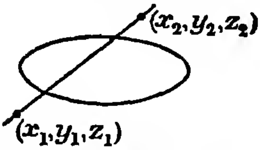


FIG 36.—INTERSECTIONS OF A LINE WITH A CONIC

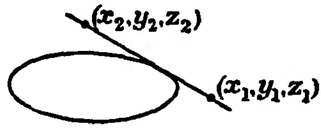


FIG 37.—TANGENT LINE TO A CONIC.

only one point of intersection arises but we say that there are two *coincident intersections* at the point; and we call the line a *tangent* at the point (Fig. 37). (This definition of tangent will be refined in section 60 (iii).)

If the fundamental quadratic in λ is satisfied identically, the whole line is part of the conic, a necessary and sufficient condition for this is that the conic should contain at least three points of the line In this case, let the line have equation $lx + my + nz = 0$. Regarding $lx + my + nz$ and S as polynomials in x, we have by ordinary division an identity of the form

$$S \equiv (lx + my + nz)(l'x + m'y + n'z) + \phi(y, z)$$

where ϕ is a homogeneous quadratic in y, z and therefore expressible as a product of two linear factors, say

$$\phi(y, z) \equiv (py + qz)(p'y + q'z).$$

Since, in the present circumstance, S vanishes for all x, y, z which annul $lx + my + nz$, the same is true of $\phi(y, z)$; but this is possible if and only if $\phi \equiv 0$. Hence, the conic is reducible.

If the conic consists of two different lines, meeting at a point O, then every line through O has two coincident intersections with the conic at O and is a tangent there. At every other point on the conic, the tangent is that line of the conic which contains the point.

If the conic consists of a single line L counted twice, then every line has two coincident intersections with the conic and is a tangent at the point where it meets L.

(iii) A reducibility condition.—We prove that a necessary and sufficient condition that the conic $S = 0$ should be reducible is that

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

First, we observe that, if the conic is reducible, there is at least one point, say (x_1, y_1, z_1) such that every line through the point has two coincident intersections with the conic there. Hence, for all x_2, y_2, z_2 the fundamental quadratic has two zero roots, thus, in particular, $S_{12} = 0$ whatever the values of x_2, y_2, z_2. Therefore

$$\begin{aligned} ax_1 + hy_1 + gz_1 &= 0, \\ hx_1 + by_1 + fz_1 &= 0, \\ gx_1 + fy_1 + cz_1 &= 0. \end{aligned}$$

Since x_1, y_1, z_1 are not all zero, we have therefore $\Delta = 0$.

Conversely, if $\Delta = 0$, there exist x_1, y_1, z_1, not all zero, to satisfy the three equations just written; and therefore such that $S_{11} = 0$ and $S_{12} = 0$ for all x_2, y_2, z_2. Hence there is a point, (x_1, y_1, z_1), such that every line through the point meets the conic there twice. A line joining this point to any other point on the conic has three intersections with the conic and is therefore part of the conic, that is, the conic is reducible.

Other proofs may be derived from the following exercises.

Ex. 1. From the identity

$$\begin{aligned} &ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \\ &\equiv (lx + my + nz)(l'x + m'y + n'z) \end{aligned}$$

show that

$$\Delta = \tfrac{1}{8} \begin{vmatrix} ll' + ll' & lm' + ml' & ln' + nl' \\ lm' + ml' & mm' + mm' & mn' + nm' \\ ln' + nl' & nm' + mn' & nn' + nn' \end{vmatrix}.$$

Arrange this determinant as a sum of determinants each of which is zero through having two columns proportional.

Ex. 2. From $\Delta = 0$, we infer $ax_1 + hy_1 + gz_1 = 0$, $hx_1 + by_1 + fz_1 = 0$, $gx_1 + fy_1 + cz_1 = 0$ for some x_1, y_1, z_1 not all zero. Suppose that $x_1 \neq 0$ and show that then

$$x_1^2 S \equiv b(xy_1 - x_1y)^2 + 2f(xy_1 - x_1y)(xz_1 - x_1z) + c(xz_1 - x_1z)^2,$$

the right-hand expression having two homogeneous linear factors.

Ex. 3. Show that

$$aS \equiv (ax + hy + gz)^2 + (Cy^2 - 2Fyz + Bz^2)$$

where $B = ca - g^2$, $F = gh - af$, $C = ab - h^2$. Hence show that, if $a \neq 0$, a necessary and sufficient condition for S to factorise is that $BC - F^2 = 0$, and verify that $BC - F^2 = a\Delta$.

Ex. 4. If $a = b = c = 0$, $S \equiv 2fyz + 2gzx + 2hxy$ and $\Delta \equiv 2fgh$ Verify that, in this case, $\Delta = 0$ is a necessary and sufficient condition for S to factorise.

Ex 5 A necessary and sufficient condition for the conic $S = 0$ to consist of a single line counted twice, is that the matrix

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

should have rank 1.

(iv) Tangent and polar lines.

(*a*) Let the conic $S = 0$ be irreducible and let (x_1, y_1, z_1) be any point on it Since $S_{11} = 0$, the fundamental quadratic has one zero root, the other root is also zero if and only if $S_{12} = 0$, that is if and only if the point (x_2, y_2, z_2) lies on the line

$$S_1 \equiv \tfrac{1}{2}\left(x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + z_1\frac{\partial}{\partial z}\right)S = 0.$$

Thus, at every point on the conic there is just one tangent line, given by $S_1 = 0$. The line is said to *touch* the conic at the point; and the point is called the *point of contact* of the tangent.

(*b*) The tangent at a point (x_1, y_1, z_1) passes through the point (x_2, y_2, z_2), not on the conic, if and only if $S_{11} = S_{12} = 0$, that

FIG 38 —POLAR LINE OF A POINT

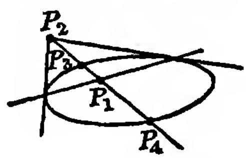


FIG 39 —HARMONIC PROPERTY OF A POINT AND ITS POLAR LINE.

is if and only if the point (x_1, y_1, z_1) is at an intersection of the conic with the line $S_2 = 0$ The line $S_2 = 0$ is not a tangent to the conic; this may be seen by showing that, if the line were tangent at (x_0, y_0, z_0), we should have $x_0 : y_0 : z_0 = x_2 : y_2 : z_2$, which is impossible for a point on the conic. Hence, two distinct tangents pass through (x_2, y_2, z_2), their points of contact lying on the line $S_2 = 0$, which is called the *polar line* of the point (x_2, y_2, z_2) (Fig. 38).

Polar lines are defined in this way only for points which are

not on the conic. In the case of a point (x_2, y_2, z_2) on the conic, the polar line is defined to be the tangent at the point and is thus again given by the equation $S_2 = 0$.

(c) The co-ordinates of all points (x_1, y_1, z_1) on either tangent from the point (x_2, y_2, z_2), which is not on the conic, are such as to make the fundamental quadratic have equal roots, that is, are such that $S_{11}S_{22} = S_{12}^2$; and *vice versa*. The locus of these points is therefore the conic $SS_{22} = S_2^2$; this conic is reducible, containing both tangents and therefore wholly composed of these tangents. We refer to $SS_{22} = S_2^2$ as the equation of the pair of tangents from (x_2, y_2, z_2).

(d) From the symmetry of the expression S_{12} with respect to the suffixes 1, 2, we see at once that if the polar of one point P contains a second point Q, then the polar of Q contains P. Two such points are said to be *conjugate* with respect to the conic, their co-ordinates are connected by the equality $S_{12} = 0$.

(e) We now prove the important harmonic relation that if a line through a point P_2 meets the polar of P_2 in P_1 and the conic in P_3, P_4, then P_1, P_2 *harm* P_3, P_4 (Fig. 39).

Taking P_1, P_2 to have co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively, we have $S_{12} = 0$. Then, if $P_3 \equiv P_1 + \lambda_3 P_2$, $P_4 \equiv P_1 + \lambda_4 P_2$, the fundamental quadratic shows that $\lambda_3 + \lambda_4 = 0$, which is the same as $\{0, \infty; \lambda_3, \lambda_4\} = -1$ Since 0 and ∞ are the parameters of P_1, P_2 respectively, we have, therefore, $\{P_1, P_2, P_3, P_4\} = -1$, which is the relation in question.

(f) In accordance with the harmonic relation just proved, we define the polar line of a point P, not on a reducible conic consisting of two different lines L, M intersecting at O, with respect to this conic to be the harmonic conjugate of OP with respect to L, M. All points on OP have then the same polar line. It is left to the reader to verify that if P has co-ordinates (x_1, y_1, z_1) and the conic has equation $S = 0$, then the polar of P is still given by $S_1 = 0$, and that if $S \equiv LM$, where $L = 0$, $M = 0$ are the equations of L, M, then $2S_1 \equiv LM_1 + L_1M$.

Similarly, in the case of a reducible conic consisting of a line L, counted twice, we define the polar of any point P to be the line L itself. The equation of the conic being $S \equiv L^2 = 0$, the polar is still given by $S_1 \equiv LL_1 = 0$, that is by $L = 0$.

16. Conic-envelopes.

In this section we consider the dual configurations corresponding to those in section 15. Since the algebra involved is the same,

but in terms of line-co-ordinates instead of point-co-ordinates, we can abbreviate.

(i) Definition.—The set of lines whose co-ordinates, in any homogeneous system, satisfy an equation of the form

$$\Sigma \equiv pl^2 + qm^2 + rn^2 + 2umn + 2vnl + 2wlm = 0,$$

where the coefficients $p, \ldots, w$ are not all zero and belong to the field of complex numbers, is called a *conic-envelope*.

The conic-envelope is said to be *irreducible* if Σ does not factorise; and to be *reducible* if Σ does factorise, in which case the conic-envelope consists of two different pencils of lines or of a single pencil of lines counted twice.

When $\Sigma = 0$ represents two different pencils of lines, we sometimes refer to the equation as being the line-equation of the pair of points which are the vertices of the two pencils. If these points have co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2), Σ has the form $k(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2)$, where k is a constant.

(ii) The fundamental quadratic.—Every line in the pencil determined by two distinct lines (l_1, m_1, n_1), (l_2, m_2, n_2) has co-ordinates of the form $(l_1 + \lambda l_2, m_1 + \lambda m_2, n_1 + \lambda n_2)$. Those lines of the pencil which belong also to the conic-envelope have parameters determined by the *fundamental quadratic*

$$\Sigma_{11} + 2\lambda\Sigma_{12} + \lambda^2\Sigma_{22} = 0,$$

where

$$\begin{aligned}
\Sigma_{11} &\equiv pl_1{}^2 + qm_1{}^2 + rn_1{}^2 + 2um_1n_1 + 2vn_1l_1 + 2wl_1m_1,\\
\Sigma_{12} &\equiv pl_1l_2 + qm_1m_2 + rn_1n_2\\
&\qquad + u(m_1n_2 + m_2n_1) + v(n_1l_2 + n_2l_1) + w(l_1m_2 + l_2m_1),\\
\Sigma_{22} &\equiv pl_2{}^2 + qm_2{}^2 + rn_2{}^2 + 2um_2n_2 + 2vn_2l_2 + 2wl_2m_2.
\end{aligned}$$

In general, therefore, every pencil contains two distinct lines of the conic-envelope. If the fundamental quadratic has equal roots, we say that there are two *coincident common lines*. If the fundamental quadratic is satisfied identically, the whole pencil is part of the conic-envelope, which is then reducible.

(iii) A reducibility condition.—A necessary condition that $\Sigma = 0$ should be reducible is that the matrix

$$\begin{pmatrix} p & w & v \\ w & q & u \\ v & u & r \end{pmatrix}$$

should have rank 2 or 1; and a necessary and sufficient condition

that $\Sigma = 0$ should be a pencil, counted twice, is that this matrix should have rank 1.

(iv) Contact points and poles.

(*a*) Let the conic-envelope $\Sigma = 0$ be irreducible and let (l_1, m_1, n_1) be any one of its lines. Since $\Sigma_{11} = 0$, the fundamental quadratic has one zero root, the other root is also zero if and only if $\Sigma_{12} = 0$, that is if and only if the line (l_2, m_2, n_2) passes through the point

$$\Sigma_1 \equiv \tfrac{1}{2}\left(l_1 \frac{\partial}{\partial l} + m_1 \frac{\partial}{\partial m} + n_1 \frac{\partial}{\partial n}\right)\Sigma = 0.$$

Thus, on every line of the conic-envelope there is just one point through which pass two coincident lines of the conic-envelope. This point is called the *contact-point* of the line in question (Fig. 40).

(*b*) The contact-point on a line (l_1, m_1, n_1) lies on the line (l_2, m_2, n_2), not belonging to the conic-envelope, if and only if $\Sigma_{11} = \Sigma_{12} = 0$, that is if, and only if, the line (l_1, m_1, n_1) is one of the two distinct lines of the conic-envelope which belong to the

FIG 40.—CONTACT-POINT OF A LINE BELONGING TO A CONIC-ENVELOPE

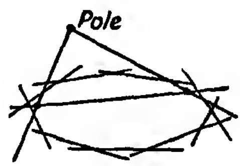


FIG 41.—POLE OF A LINE

pencil $\Sigma_2 = 0$. Thus, on every line, not belonging to the conic-envelope, there are two contact-points; the vertex of the pencil $\Sigma_2 = 0$ is called the *pole* of the line (l_2, m_2, n_2) (Fig 41).

The pole of a line belonging to the conic-envelope is defined to be the contact-point on the line.

(*c*) The line-equation of the pair of contact-points on the line (l_2, m_2, n_2), which does not belong to the conic-envelope, is $\Sigma\Sigma_{22} = \Sigma_2^{\,2}$.

(*d*) If the pole of a line L lies on a second line M, then the pole of M lies on L. Two such lines are said to be *conjugate* with respect to the conic envelope, their co-ordinates are connected by the equation $\Sigma_{12} = 0$.

(*e*) Let P be any point on a given line L_1 and let L_2 be the line joining P to the pole of L_1; also let L_3, L_4 be the lines of the conic-envelope which pass through P; then L_1, L_2 *harm* L_3, L_4.

(*f*) In accordance with the harmonic relation just stated, we define the pole of a line L with respect to a reducible conic-envelope, consisting of two pencils with distinct vertices A, B, to be the harmonic conjugate with respect to A, B of the point where L meets AB. It may be verified that if L has co-ordinates (l_1, m_1, n_1) and the conic-envelope is given by $\Sigma \equiv AB = 0$, then the line-equation of the pole is $2\Sigma_1 \equiv AB_1 + A_1B = 0$.

Similarly, the pole of any line L with respect to a conic-envelope, which consists of a pencil counted twice, is defined to be the vertex of the pencil.

17. The conic-envelope associated with a conic, and *vice versa*.

(*a*) In what follows, $A, \ldots, H$ denote the co-factors of $a, \ldots, h$ in the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

The determinant

$$\Delta^* \equiv \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

is equal to Δ^2 and therefore is zero if and only if Δ is zero. The co-factors of $A, \ldots, H$ in Δ^* are respectively $a\Delta, \ldots, h\Delta$.

(*b*) The tangent at a point (x_1, y_1, z_1) to the irreducible conic $S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has co-ordinates (l_1, m_1, n_1) such that

$$\begin{aligned} ax_1 + hy_1 + gz_1 &= \rho l_1, \\ hx_1 + by_1 + fz_1 &= \rho m_1, \\ gx_1 + fy_1 + cz_1 &= \rho n_1, \end{aligned}$$

for some $\rho \neq 0$; and

$$l_1x_1 + m_1y_1 + n_1z_1 = 0$$

Eliminating x_1, y_1, z_1, ρ, we find that the tangent (l_1, m_1, n_1) belongs to the conic-envelope

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0$$

which is the same as

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

and is irreducible.

(*c*) Conversely, the contact-point of any line (l_1, m_1, n_1) belonging to $\Sigma = 0$ has co-ordinates (x_1, y_1, z_1) such that

$$\begin{aligned} Al_1 + Hm_1 + Gn_1 &= \sigma x_1, \\ Hl_1 + Bm_1 + Fn_1 &= \sigma y_1, \\ Gl_1 + Fm_1 + Cn_1 &= \sigma z_1, \end{aligned}$$

for some $\sigma \neq 0$; and

$$x_1 l_1 + y_1 m_1 + z_1 n_1 = 0.$$

Therefore, the contact-point (x_1, y_1, z_1) is on the conic

$$\Delta S \equiv \begin{vmatrix} A & H & G & x \\ H & B & F & y \\ G & F & C & z \\ x & y & z & 0 \end{vmatrix} = 0, \text{ i.e. } S = 0.$$

(*d*) We note further, in regard to the two sets of four linear equations, that if we solve the first three equations of the first set for x_1, y_1, z_1 we obtain the first three equations of the second set, and *vice versa*. Therefore every point of $S = 0$ is a contact-point of $\Sigma = 0$, and *vice versa*; and every line of $\Sigma = 0$ is a tangent line of $S = 0$, and *vice versa*. Thus, $\Sigma = 0$ is the set of tangent lines of $S = 0$; and $S = 0$ is the set of contact-points of $\Sigma = 0$. We say that $S = 0$ and $\Sigma = 0$ are *associated* (Fig. 42).

FIG 42.—ASSOCIATED CONIC AND CONIC-ENVELOPE.

(*e*) By re-interpreting the first three linear equations in each of the above sets, we see at once that if (l_1, m_1, n_1) is the polar of (x_1, y_1, z_1) with respect to $S = 0$, then (x_1, y_1, z_1) is the pole of (l_1, m_1, n_1) with respect to $\Sigma = 0$, and *vice versa*. It is convenient to say that the point and line are pole and polar with respect to both the conic and the conic-envelope.

Ex. 1. If $S = 0$ represents a pair of distinct lines, $\Sigma = 0$ is the pencil, counted twice, with vertex at the common point of these lines. If $S = 0$ represents a line, counted twice, then $\Sigma \equiv 0$.

Ex. 2. If $\Sigma = 0$ represents a pair of distinct pencils, $S = 0$ represents their common line, counted twice. If $\Sigma = 0$ represents a pencil, counted twice, then $S \equiv 0$.

18. Special forms for the equations of a conic and of a conic-envelope.

(*a*) The equation of a conic passing through the vertices of the triangle of reference is obviously of the form

$$fyz + gzx + hxy = 0,$$

which is sometimes preferably expressed as

$$\frac{f}{x} + \frac{g}{y} + \frac{h}{z} = 0.$$

The associated conic-envelope has the equation

$$-f^2l^2 - g^2m^2 - h^2n^2 + 2ghmn + 2hfnl + 2fglm = 0.$$

Dually, the equation of a conic-envelope, to which belong the sides of the triangle of reference, has the form

$$Fmn + Gnl + Hlm = 0$$

or

$$\frac{F}{l} + \frac{G}{m} + \frac{H}{n} = 0;$$

and the associated conic has the equation

$$-F^2x^2 - G^2y^2 - H^2z^2 + 2GHyz + 2HFzx + 2FGxy = 0.$$

More generally, the equation of a conic passing through the vertices of the triangle formed by three lines whose equations are $p = 0$, $q = 0$, $r = 0$, where p, q, r are linear in x, y, z, has the form

$$fqr + grp + hpq = 0;$$

and the equation of a conic-envelope containing the sides of a triangle whose vertices have the line-equations $u = 0$, $v = 0$, $w = 0$, where u, v, w are linear in l, m, n, has the form

$$Fvw + Gwu + Huv = 0.$$

(*b*) Let A be any point not on an irreducible conic. On its polar line A we take any point B, not on the conic. The polar line B of B passes through A and meets A in a point C, whose polar line C is AB. Such a triangle is said to be *self-polar* or *self-conjugate* with respect to the conic (Fig 43).

Taking ABC as triangle of reference, the equation of the conic has the form

$$ax^2 + by^2 + cz^2 = 0, \text{ with } abc \neq 0;$$

and the equation of the associated conic-envelope is

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0.$$

By choosing the unit point or line suitably, these equations become

$$x^2 + y^2 + z^2 = 0$$

and
$$l^2 + m^2 + n^2 = 0.$$

More generally, the equation of a conic, which has a self-polar triangle whose sides are given by $p = 0$, $q = 0$, $r = 0$, has the form

$$ap^2 + bq^2 + cr^2 = 0;$$

and the equation of a conic-envelope, which has a self-polar triangle whose vertices are given by $u = 0$, $v = 0$, $w = 0$, has the form

$$Au^2 + Bv^2 + Cw^2 = 0.$$

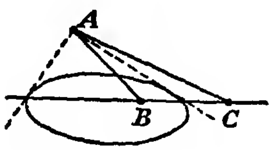


FIG. 43.—SELF-POLAR TRIANGLE

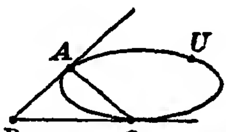


FIG 44.—FRAME OF REFERENCE FOR THE EQUATION $y^2 - zx = 0$.

Ex. 1. ABC and $A'B'C'$ are two triangles, each self-polar with respect to a given conic. Prove that there exists a conic with respect to which $AB'C'$ and $A'BC$ are two self-polar triangles

(c) Let the tangents at two distinct points A, C of an irreducible conic meet at B (Fig. 44). With ABC as triangle of reference, the equation of the conic has the form

$$by^2 + 2gzx = 0;$$

and if the unit point is taken to be a point on the conic, the equation is further simplified to

$$zx - y^2 = 0.$$

Dually, let the contact-points of two distinct lines A, C of an irreducible conic envelope be joined by B. With ABC as triangle of reference, the equation of the conic-envelope has the form

$$Bm^2 + 2Gnl = 0;$$

and if the unit line is taken to be a line of the conic-envelope, the equation becomes

$$nl - m^2 = 0.$$

More generally, if an irreducible conic touches the lines $p = 0$, $r = 0$ at points on the line $q = 0$, its equation has the form

$rp + kq^2 = 0$; and if an irreducible conic-envelope contains two lines whose contact points are given by $u = 0$, $w = 0$, and these lines meet in the point given by $v = 0$, its equation has the form $wu + hv^2 = 0$.

Ex. 2. The conic-envelope associated with $zx - y^2 = 0$ is $4nl - m^2 = 0$, and the conic associated with $nl - m^2 = 0$ is $4zx - y^2 = 0$

Ex 3. A conic touches the sides AB, AC of a triangle ABC at B, C respectively and the sides $A'B'$, $A'C'$ of another triangle at B', C' respectively. Prove that there exists a conic which touches AB, AC where they are met by $B'C'$ and touches $A'B'$, $A'C'$ where they are met by BC.

19. Miscellaneous theorems.

(i) Hesse's theorem.—We prove that if two pairs of opposite vertices of a quadrilateral are pairs of conjugate points with respect to a conic, then the third pair of opposite vertices is a pair of conjugate points

We take the diagonal triangle of the quadrilateral ABCD as triangle of reference and A as unit line. Then the four lines have co-ordinates

$$\mathsf{A}(1, 1, 1),\ \mathsf{B}(-1, 1, 1),\ \mathsf{C}(1, -1, 1),\ \mathsf{D}(1, 1, -1);$$

and the pairs of opposite vertices have co-ordinates

$$\begin{array}{lll} \mathsf{A}.\mathsf{B}\ (0, -1, 1), & \mathsf{A}.\mathsf{C}\ (1, 0, -1), & \mathsf{A}.\mathsf{D}\ (-1, 1, 0), \\ \mathsf{C}\ \mathsf{D}\ (0, 1, 1), & \mathsf{D}.\mathsf{B}\ (1, 0, 1), & \mathsf{B}.\mathsf{C}\ (1, 1, 0). \end{array}$$

Let the equation of the conic be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Then, $\mathsf{A}.\mathsf{B}$ and $\mathsf{C}.\mathsf{D}$ are conjugate if and only if $b = c$; and $\mathsf{A}.\mathsf{C}$ and $\mathsf{D}.\mathsf{B}$ are conjugate if and only if $c = a$. If, then, both pairs are conjugate pairs, we have $a = b$, so that $\mathsf{A}.\mathsf{D}$ and $\mathsf{B}.\mathsf{C}$ are also conjugate.

Ex. 1. Prove the dual theorem that if two pairs of opposite sides of a quadrangle are pairs of conjugate lines with respect to a conic-envelope, the third pair of sides is also a pair of conjugate lines.

(ii) Polar triangles are in perspective.—Let ABC be any triangle which is not self-polar with respect to an irreducible conic $S = 0$. The polar lines of A, B, C form a triangle $A'B'C'$, called the *polar triangle* of ABC with respect to the conic. Clearly ABC is the polar triangle of $A'B'C'$. We prove that, $B'C'$ being the polar of A, and so on, the triangles ABC, $A'B'C'$ are in perspective.

We take A to have co-ordinates (x_1, y_1, z_1) and so on; and denote the expressions

$$\tfrac{1}{2}\frac{\partial S_{11}}{\partial x_1}, \quad \tfrac{1}{2}\frac{\partial S_{11}}{\partial y_1}, \quad \tfrac{1}{2}\frac{\partial S_{11}}{\partial z_1}$$

by X_1, Y_1, Z_1 respectively, and so on. Then $C'A'$ has the equation $xX_2 + yY_2 + zZ_2 = 0$, and $A'B'$ has the equation $xX_3 + yY_3 + zZ_3 = 0$. Hence the equation of AA' is of the form

$$p(xX_2 + yY_2 + zZ_2) + q(xX_3 + yY_3 + zZ_3) = 0$$

with

$$p(x_1X_2 + y_1Y_2 + z_1Z_2) + q(x_1X_3 + y_1Y_3 + z_1Z_3) = 0,$$

that is with

$$pS_{12} + qS_{13} = 0.$$

Therefore AA' has the equation

$$x(S_{13}X_2 - S_{12}X_3) + y(S_{13}Y_2 - S_{12}Y_3) + z(S_{13}Z_2 - S_{12}Z_3) = 0.$$

Similarly, BB' and CC' have respectively the equations

$$x(S_{21}X_3 - S_{23}X_1) + y(S_{21}Y_3 - S_{23}Y_1) + z(S_{21}Z_3 - S_{23}Z_1) = 0,$$
$$x(S_{32}X_1 - S_{31}X_2) + y(S_{32}Y_1 - S_{31}Y_2) + z(S_{32}Z_1 - S_{31}Z_2) = 0$$

The left-hand sides of these three equations add up to zero hence, the three lines AA', BB', CC' are concurrent.

The algebra is simpler if ABC is taken as triangle of reference.

(iii) Six lines belonging to a conic-envelope.—We prove that if every intersection of an irreducible conic with a side of a triangle, in general position relative to the conic, is joined to the vertex opposite to that side, then the six lines so obtained belong to a conic-envelope

Let the triangle ABC be taken as triangle of reference; and let the equation of the conic be $S = 0$.

The two lines passing through A are given by

$$by^2 + 2fyz + cz^2 = 0.$$

If they have line-co-ordinates $(0, m_1, n_1)$ and $(0, m_2, n_2)$ then $-n_1/m_1$ and $-n_2/m_2$ are the two values of y/z given by $by^2 + 2fyz + cz^2 = 0$. Therefore the pairs of numbers m_1, n_1 and m_2, n_2 each satisfy the equation in m, n

$$bn^2 + 2fnm + cm^2 = 0$$

or

$$\frac{m^2}{b} + \frac{2fmn}{bc} + \frac{n^2}{c} = 0.$$

The two lines in question therefore belong to the conic-envelope

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}+\frac{2fmn}{bc}+\frac{2gnl}{ca}+\frac{2hlm}{ab}=0;$$

and, similarly, the other two pairs of lines also belong to it.

Ex. 2. State and prove the dual theorem.

(iv) Hessian point and line.—Let ABC be a triangle whose sides BC, CA, AB touch an irreducible conic at the points A', B', C' respectively. We prove that AA', BB', CC' are concurrent at a point H, called the *Hessian point* of the three tangents, and that the points $BC\,.\,B'C'$, $CA\,.\,C'A'$, $AB\,.\,A'B'$ are on a line H, called the *Hessian line* of A', B', C' (Fig. 45).

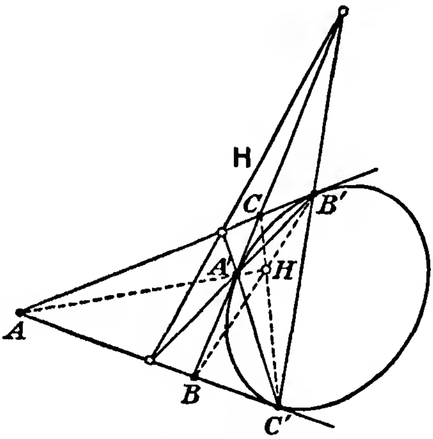


FIG 45.—HESSIAN POINT AND LINE.

With ABC as triangle of reference, the equation of the associated conic-envelope has the form

$$Fmn + Gnl + Hlm = 0.$$

Hence, A', as pole or contact-point of the line $(1, 0, 0)$, has co-ordinates $(0, G^{-1}, H^{-1})$. Thus AA' and, similarly, BB' and CC' pass through the point (F^{-1}, G^{-1}, H^{-1}).

The second part of the theorem comes immediately by considering the harmonic polar line of H relative to the triangle ABC, H has co-ordinates (F, G, H).

Clearly, H and H are respectively pole and polar line relative to the given conic. The triangles are in perspective with centre H and axis H.

The figure is related to several which have been mentioned already, as is indicated in exercises which follow.

Ex 3. Prove the above theorem, taking $A'B'C'$ as triangle of reference.

Ex. 4. Prove the converse theorem.

Ex. 5. Deduce the theorem from part (ii) of this section.

Ex. 6. The theorem follows from Hesse's theorem for a quadrilateral or from the dual theorem for a quadrangle. [Thus,

let $D' = BB' . CC'$; then $A'C'$, $D'B'$ are conjugate; and so are $A'B'$, $C'D'$. Therefore $A'D'$, $B'C'$ are conjugate and consequently $A'D'$ contains the pole A of $B'C'$.]

Ex 7. In part (iii) of this section let the conic $S = 0$ approach a limiting position which is a conic inscribed in the triangle ABC. Show that the limiting position of the derived conic-envelope is the pencil, counted twice, with vertex at the Hessian point.

(v) Quadrilateral of tangents at the vertices of a quadrangle inscribed in a conic.—Let A, B, C, D be four distinct points on an irreducible conic; and let A, B, C, D be the tangents at these points. The quadrangle $ABCD$ and the quadrilateral ABCD are connected as follows in several interesting ways.

Let $E = AB$ CD, $F = AC . DB$, $G = AD . BC$ be the diagonal points of $ABCD$. EFG is a triangle self-polar for the conic. Similarly, let E, F, G be the diagonals of ABCD. We prove that E is FG, F is GE, G is EF

Taking EFG as triangle of reference and A as unit point, the conic has an equation of the form $ax^2 + by^2 + cz^2 = 0$, with $a + b + c = 0$; and A, B, C, D have co-ordinates (a, b, c), $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$ respectively. Hence, the opposite vertices A . B and C . D of the quadrilateral have respectively co-ordinates $(0, -b^{-1}, c^{-1})$ and $(0, b^{-1}, c^{-1})$; these lie on FG and are harmonic with respect to F, G The theorem therefore follows, with the addition of these harmonic properties.

Ex. 8. The opposite sides AB, CD of the quadrangle are harmonic with respect to F, G; and similarly in regard to the other pairs of opposite sides

20. The projective generation of a conic and of a conic-envelope. Parametric representation.

(*a*) We have seen that the equation of an irreducible conic may be given the form

$$xz - y^2 = 0$$

relative to a triangle of reference ABC, with AB, CB touching the conic at A, C respectively, and to a unit point on the conic

The line $y - \lambda z = 0$ meets the conic at the point A and again at the point P with co-ordinates $(\lambda^2, \lambda, 1)$, and the line joining P to C has the equation $x - \lambda y = 0$.

As λ varies throughout the modified system of complex numbers, the lines AP, CP vary throughout the pencils with vertices at A, C and correspond in a projectivity. The conic is said to be *generated projectively* by means of the two pencils (Fig. 46). Evidently, such a projective generation may be associated with every pair of distinct points on the conic.

We call λ the *parameter* of the point P relative to the selected frame of reference. If we assign to A, B, C respectively the co-ordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ then

$$P \equiv \lambda^2 A + \lambda B + C.$$

The parameters of A, C are ∞, 0 respectively. AC, as a line through A, corresponds to CB; and AB corresponds to CA, as

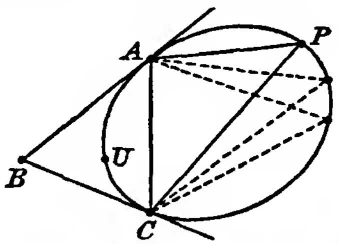


FIG. 46.—PROJECTIVE GENERATION OF A CONIC.

a line through C; that is the tangents to the conic at A, C are the lines in each pencil which correspond to AC in the other.

Ex. 1. Discuss the parametric representation of the conic-envelope $nl - m^2 = 0$, and show that the lines of the conic-envelope meet the lines $(1, 0, 0)$ and $(0, 0, 1)$ in points which correspond in a projectivity; show also that the contact-points of these lines are the points which correspond on each line to their common point (Fig. 47).

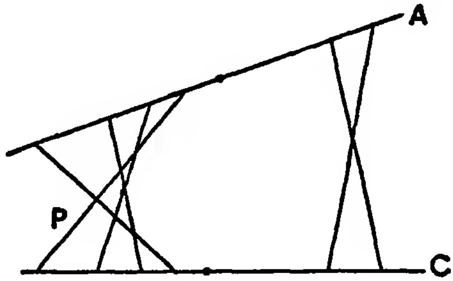


FIG. 47 —PROJECTIVE GENERATION OF A CONIC-ENVELOPE.

(*b*) Conversely, let us consider any projectivity between two pencils of lines and investigate the locus of the point in which corresponding lines meet.

Let the vertices of the two pencils be A, C and let the equation of AC be $v = 0$. Let $u = 0$ be a different line through A, and $w = 0$ be the corresponding line through C. Then let the line

$u - \lambda v = 0$ through A correspond to the line $v - \mu w = 0$ through C.

The equation of the correspondence must have the form $a\lambda\mu + b\lambda + d = 0$, with $ad \neq 0$, since $\mu = \infty$ when $\lambda = 0$. The locus of P is therefore the conic $auv + buw + dvw = 0$. It is reducible if and only if $b = 0$, in which case AC is a self-corresponding line and the locus consists of the two lines $v = 0$, $au + dw = 0$.

Let (x_1, y_1, z_1) be a triad of co-ordinates for A, and let u_1 be the result of substituting x_1, y_1, z_1 for x, y, z in u, and so on. In the general case ($b \neq 0$), the conic passes through A, where $u = v = 0$, and C, where $v = w = 0$. The equation of the tangent at A is $a(uv_1 + u_1v) + b(uw_1 + u_1w) + d(vw_1 + v_1w) = 0$, that is $bu + dv = 0$, since $u_1 = v_1 = 0$. This is the line through A which corresponds to CA through C. Similarly, the tangent at C is $av + bw = 0$, which is the line through C corresponding to AC through A.

Ex 2. Investigate in the same way the conic-envelope generated by the line joining corresponding points in a projectivity between two lines.

Ex. 3. $PP_1 \ldots P_n$ is a variable polygon whose vertices $P_1, P_2, \ldots, P_n$ lie on fixed lines and all of whose sides pass through fixed points. Prove that the locus of P is a conic passing through the fixed points on PP_1, PP_n. State the dual theorem.

(*c*) The co-ordinates of a variable point on an irreducible conic have been expressed by means of the *parametric equations*

$$x : y : z = \lambda^2 : \lambda : 1$$

relative to a particular type of frame of reference. If (X, Y, Z) are co-ordinates of the point relative to any other frame of reference, these are connected with (x, y, z) by equations of the form

$$X : Y : Z = a_1x + a_2y + a_3z \;.\; b_1x + b_2y + b_3z : c_1x + c_2y + c_3z,$$

where

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

Relative to the new frame of reference we have the parametric equations

$$X : Y : Z = a_1\lambda^2 + a_2\lambda + a_3 : b_1\lambda^2 + b_2\lambda + b_3 : c_1\lambda^2 + c_2\lambda + c_3.$$

Ex. 4. Discuss the position of the triangle of reference relative to the conic given by $x : y : z = \lambda^2 : 2 - \lambda^2 : \lambda$.

Ex. 5. A change of parameter to μ, where $\mu = (p\lambda + q)/(r\lambda + s)$, with $ps \neq qr$, leaves the general form of the

parametric equations unaltered. Such a transformation is convenient when we wish to assign particular parameters, usually $0, 1, \infty$, to three particular points on the conic.

Ex. 6 In the general case considered above, the parametric equations for the associated conic-envelope are

$$L : M : N$$
$$= A_1 - 2A_2\lambda + A_3\lambda^2 : B_1 - 2B_2\lambda + B_3\lambda^2 : C_1 - 2C_2\lambda + C_3\lambda^2,$$

where A_1 is the co-factor of a_1 in Δ, and so on.

21. A cross-ratio property of a conic.

(*a*) We consider an irreducible conic and take its equation in the form $xz - y^2 = 0$ or, parametrically, $x : y : z = \lambda^2 . \lambda : 1$.

The equation of the chord joining two points, with parameters θ, ϕ, is easily seen to be

$$x - y(\theta + \phi) + z\theta\phi = 0;$$

and the equation of the tangent at the point with parameter θ is

$$x - 2y\theta + z\theta^2 = 0.$$

The linearity of the equation of the chord with respect to each parameter has several important consequences. For the present, we observe that if P, P_1, P_2, P_3, P_4 are five distinct points on the conic, with parameters λ, λ_1, λ_2, λ_3, λ_4 respectively, the equations of the four chords PP_i are $(x - \lambda y) - \lambda_i(y - \lambda z) = 0$, $i = 1, 2, 3, 4$. Hence, $P\{P_1, P_2; P_3, P_4\} = \{\lambda_1, \lambda_2; \lambda_3, \lambda_4\}$, which is independent of the position of P on the conic, provided that it does not coincide with any one of the points P_i. The relation clearly continues to hold in the exceptional circumstances provided that we make the convention (to be adhered to) that, when P is at P_i, PP_i is to mean the tangent at P_i

It follows that, if Q is any other point on the conic, there is a projectivity between the pencils with vertices at P, Q such that PP_i corresponds to QP_i, $i = 1, 2, 3, 4$

It is convenient to call $\{\lambda_1, \lambda_2; \lambda_3, \lambda_4\}$ a *cross-ratio* of the four points P_1, P_2, P_3, P_4 *on the conic* and to denote it by $\{P_1, P_2, P_3, P_4\}$. It will be understood that the latter symbol depends on the conic as well as on the four points

(*b*) Conversely, we show that if P is a variable point and P_1, P_2, P_3, P_4 are four fixed points, of which no three are in line, such that $P\{P_1, P_2, P_3, P_4\} = k$, a constant, then the locus of P is a conic through P_1, P_2, P_3, P_4.

We may take the points P, P_1, P_2, P_3, P_4 to have co-ordinates (x, y, z), $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ respectively. Then PP_1, PP_2, PP_3, PP_4 have co-ordinates

$(y - z, \quad z - x, \quad x - y)$, $(y - z, \quad -z - x, \quad x + y)$, $(y + z,\ z - x,\ -x - y)$, $(-y - z,\ z + x,\ x - y)$ respectively; and thus

$$(y - z)PP_3 \equiv (x + y)PP_1 + (z - x)PP_2,$$
$$(z - y)PP_4 \equiv (z + x)PP_1 + (y - x)PP_2.$$

Hence $\quad k = P\{P_1, P_2;\ P_3, P_4\}$

$$= \{0, \infty;\ (z - x)(x + y)^{-1},\ (y - x)(z + x)^{-1}\};$$

and therefore

$$(1 - k)x^2 + ky^2 - z^2 = 0.$$

The locus of P is therefore a conic, obviously passing through each of P_1, P_2, P_3, P_4. It is reducible if and only if $k = 0$, 1 or ∞, and is then a pair of opposite sides of the quadrangle $P_1P_2P_3P_4$.

Ex. 1. State and prove the corresponding properties of a conic-envelope.

Ex. 2. The diagonal triangle of $P_1P_2P_3P_4$ is self-polar with respect to the conic which is the locus of P.

22. The theorems of Pascal and Brianchon.

(i) Pascal's theorem.

(*a*) Let A, B, C, A', B', C' be six different points on an irreducible conic. Pascal's theorem is that the points $P = BC' . B'C$, $Q = CA' . C'A$, $R = AB' . A'B$ are collinear (Fig. 48).

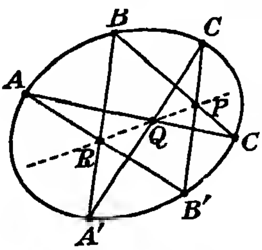


FIG 48.—FIGURE FOR PASCAL'S THEOREM

The theorem involves a resolution of the six points into two triads A, B, C and A', B', C' in such a way that the order A, B, C is associated with the order A', B', C'. We refer to the six points as determining a hexagon and indicate the resolution just mentioned by naming the hexagon by the symbol $(ABC, A'B'C')$. Then PQR is called the *Pascal line* of this hexagon, and P, Q, R are called the intersections of the pairs of *opposite sides*.

(*b*) Algebraically, let D be the pole of AA' and let us take ADA' as triangle of reference, so that DA' is $x = 0$, AA' is $y = 0$, AD is $z = 0$. Let the unit point be chosen on the conic, which may then be represented by the parametric equations $x : y : z = \lambda^2 : \lambda : 1$. The parameters of A, A' are ∞, 0 respectively; let B have parameter b, and so on.

The points P, Q, R are then easily found to have co-ordinates

$(cc'(b-b')-bb'(c-c'),\quad c'b-b'c,\quad (b-b')-(c-c'))$, $(cc', c', 1)$, $(bb', b', 1)$ respectively. We have, therefore

$$P \equiv (b-b')Q-(c-c')R;$$

consequently P, Q, R are in line.

(c) Alternatively, let $H = AB' . BC'$, $K = AC' . B'C$. Then, since $B(A, A', B', C') \barwedge C(A, A', B', C')$, we have $(A, R, B', H) \barwedge (A, Q, K, C')$. Therefore, the projectivity which transforms R, B', H on AB' into Q, K, C' on AC' has A as a self-corresponding point and is therefore a perspective Hence, RQ, $B'K$, HC' are concurrent. $B'K$ meets HC' in P; therefore P, Q, R are in line

(d) The first proof of Pascal's theorem remains valid when $B' = A$, $C' = B$, $A' = C$ provided that we interpret, as already decided, AB' as the tangent at A, and so on. In this case the Pascal line becomes the Hessian line of the triad A, B, C. And, of course, the theorem is also valid with fewer coincidences among the six points.

These special cases of the theorem are useful in a number of problems.

(e) Pascal's theorem was enunciated and proved relative to an irreducible conic In the case of a conic consisting of two distinct lines, with A, B, C on one line and A', B', C' on the other, the theorem is still true, being then Pappus' theorem.

(f) The converse of Pascal's theorem is that, if the points P, Q, R are in line and no three of A, B, C, A', B', C' are in line, then the six points last named lie on an irreducible conic.

There is, in fact, as we shall prove in section 23 (b), just one conic through A, B, C, A', B'; let it meet BC' again at C''. Then by the direct theorem $P = BC''\ B'C$, $CA' . C''A$, $R = AB'\ A'B$ are in line. Therefore CA' meets $C''A$ on PR, necessarily therefore at Q. Hence C'' is C'; and so the conic in question passes through all six points.

Ex. 1. Sixty different Pascal lines may be obtained from six given points on a conic by suitable associations between the points.

Ex. 2 The Pascal lines of the hexagons $(ABC, A'B'C')$, $(ABC, B'C'A')$, $(ABC, C'A'B')$ are concurrent.

(ii) **Brianchon's theorem.**—Let A, B, C, A′, B′, C′ be six different lines belonging to an irreducible conic-envelope (Fig. 49). Brianchon's theorem is the dual of Pascal's and asserts that the lines P joining B . C′ to B′ . C, Q joining C . A′ to C′ . A, R joining A . B′ to A′ . B are concurrent in a point called the ***Brianchon point*** of the hexalateral (ABC, A′B′C′).

The proofs of Pascal's theorem may be dualised without difficulty. Brianchon's theorem holds also when any or all of the coincidences $\mathsf{A} = \mathsf{B}'$, $\mathsf{B} = \mathsf{C}'$, $\mathsf{C} = \mathsf{A}'$ occur, provided that we mean by $\mathsf{A} \,.\, \mathsf{B}'$ the contact-point of A, and so on. When all three coincidences occur, the Brianchon point becomes the Hessian point of A, B, C.

Ex. 3. Sixty different Brianchon points may be obtained from six given lines in a conic-envelope by suitable associations between the lines

Ex. 4. The Brianchon points of the hexalaterals (ABC, $\mathsf{A'B'C'}$), (BCA, $\mathsf{A'B'C'}$), (CAB, $\mathsf{A'B'C'}$) are collinear.

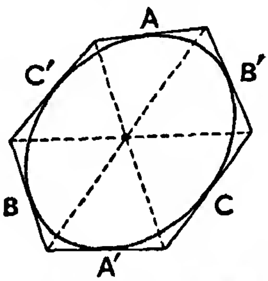


FIG 49—FIGURE FOR BRIANCHON'S THEOREM

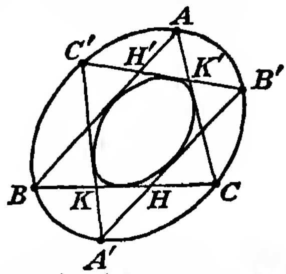


FIG 50—TRIANGLES WHOSE VERTICES LIE ON ONE CONIC AND WHOSE SIDES TOUCH ANOTHER CONIC.

(iii) Two triangles inscribed in a conic.

(*a*) We prove that, if two triangles ABC, $A'B'C'$ are inscribed in an irreducible conic and the vertices are all different, their sides belong to an irreducible conic-envelope The converse of this theorem is the same as the dual.

Let (Fig. 50) $A'B'$, $A'C'$ meet BC in H, K; and let AB, AC meet $B'C'$ in H', K'. Since the conic is irreducible, we have $A(B, C, B', C') \,\overline{\wedge}\, A'(B, C, B', C')$, and therefore, by incidence, $(H', K', B', C') \,\overline{\wedge}\, (B, C, H, K)$ We are thus led to a projectivity between $B'C'$ and BC which generates a conic-envelope containing $B'C'$, BC and $H'B$, $K'C$, $B'H$, $C'K$, that is containing the six sides of the two triangles. The conic-envelope is irreducible since no three of the six sides are concurrent.

(*b*) The algebraic proof of the above theorem is interesting and leads to new proofs of the theorems of Pascal and Brianchon.

We take ABC as triangle of reference and let $B'C'$, $C'A'$, $A'B'$ have co-ordinates (a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}) respectively. Then the equation of the given conic is

$$\frac{\lambda_1}{a_{11}x + a_{12}y + a_{13}z} + \frac{\lambda_2}{a_{21}x + a_{22}y + a_{23}z} + \frac{\lambda_3}{a_{31}x + a_{32}y + a_{33}z} = 0,$$

where λ_1, λ_2, λ_3 are all non-zero and

$$\lambda_1 a_{1i}^{-1} + \lambda_2 a_{2i}^{-1} + \lambda_3 a_{3i}^{-1} = 0, \; i = 1, 2, 3,$$

so that

$$\begin{vmatrix} a_{11}^{-1} & a_{21}^{-1} & a_{31}^{-1} \\ a_{12}^{-1} & a_{22}^{-1} & a_{32}^{-1} \\ a_{13}^{-1} & a_{23}^{-1} & a_{33}^{-1} \end{vmatrix} = 0.$$

From the three equations involving λ_1, λ_2, λ_3, or from the vanishing of the determinant, μ_1, μ_2, μ_3 exist such that

$$\mu_1 a_{i1}^{-1} + \mu_2 a_{i2}^{-1} + \mu_3 a_{i3}^{-1} = 0, \; i = 1, 2, 3,$$

and μ_1, μ_2, μ_3 are all non-zero since the lines $B'C'$, $C'A'$, $A'B'$ are different.

The last three equations show that $B'C'$, $C'A'$, $A'B'$ belong to the irreducible conic-envelope

$$\frac{\mu_1}{l} + \frac{\mu_2}{m} + \frac{\mu_3}{n} = 0,$$

which also contains BC, CA, AB

(c) Let A_{ij} denote the co-factor of a_{ij} in the determinant

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

Then A', B', C' have co-ordinates (A_{11}, A_{12}, A_{13}), (A_{21}, A_{22}, A_{23}), (A_{31}, A_{32}, A_{33}) respectively The equation of the conic-envelope may be expressed in the form

$$\frac{\theta_1}{A_{11}l + A_{12}m + A_{13}n} + \frac{\theta_2}{A_{21}l + A_{22}m + A_{23}n} + \frac{\theta_3}{A_{31}l + A_{32}m + A_{33}n} = 0,$$

with

$$\theta_1 A_{1i}^{-1} + \theta_2 A_{2i}^{-1} + \theta_3 A_{3i}^{-1} = 0, \; i = 1, 2, 3,$$

so that

$$\begin{vmatrix} A_{11}^{-1} & A_{21}^{-1} & A_{31}^{-1} \\ A_{12}^{-1} & A_{22}^{-1} & A_{32}^{-1} \\ A_{13}^{-1} & A_{23}^{-1} & A_{33}^{-1} \end{vmatrix} = 0.$$

(d) With the notation of part (i), we may verify without difficulty that the points P, Q, R, which determine the Pascal line of the hexagon $(ABC, A'B'C')$, have co-ordinates $(1, A_{22}A_{21}^{-1}, A_{33}A_{31}^{-1})$, $(A_{11}A_{12}^{-1}, 1, A_{33}A_{32}^{-1})$, $(A_{11}A_{13}^{-1}, A_{22}A_{23}^{-1}, 1)$ respectively. That these points are in line follows from the fact that

E

$$\begin{vmatrix} 1 & A_{22}A_{21}^{-1} & A_{33}A_{31}^{-1} \\ A_{11}A_{12}^{-1} & 1 & A_{33}A_{32}^{-1} \\ A_{11}A_{13}^{-1} & A_{22}A_{23}^{-1} & 1 \end{vmatrix} = A_{11}A_{22}A_{33} \begin{vmatrix} A_{11}^{-1} & A_{21}^{-1} & A_{31}^{-1} \\ A_{12}^{-1} & A_{22}^{-1} & A_{32}^{-1} \\ A_{13}^{-1} & A_{23}^{-1} & A_{33}^{-1} \end{vmatrix} = 0.$$

The equation of the Pascal line is easily seen to be

$$A_{11}^{-1}\theta_1 x + A_{22}^{-1}\theta_2 y + A_{33}^{-1}\theta_3 z = 0.$$

(*e*) Similarly, the lines P, Q, R, which determine the Brianchon point of the hexalateral ($BC\ CA\ AB$, $B'C'\ C'A'\ A'B'$), have co-ordinates $(1,\ a_{22}a_{21}^{-1},\ a_{33}a_{31}^{-1})$, $(a_{11}a_{12}^{-1},\ 1,\ a_{33}a_{32}^{-1})$, $(a_{11}a_{13}^{-1},\ a_{22}a_{23}^{-1},\ 1)$ respectively, and these are now obviously concurrent. The equation of the Brianchon point is

$$a_{11}^{-1}\lambda_1 l + a_{22}^{-1}\lambda_2 m + a_{33}^{-1}\lambda_3 n = 0.$$

(*f*) We prove that, if there is one triangle having its vertices on a given irreducible conic and its sides belonging to a given irreducible conic-envelope, then there are an infinity of such triangles.

Let ABC be the given triangle. If A' is any point on the conic other than A, B, C and the intersections of the conic with the associated conic of the conic-envelope, there pass through A' two lines of the conic-envelope; let these meet the given conic at B', C'. By the theorem of (*a*) above, the sides of the two triangles belong to an irreducible conic-envelope; this conic-envelope has five lines in common with the given conic-envelope and therefore coincides with it. Hence $B'C'$ belongs to the given conic-envelope; and the theorem follows.

This proof anticipates the results of the next section where we prove the dual of the statement that a single irreducible conic-envelope contains five given lines of which no three are concurrent.

(*g*) It is appropriate to mention now another theorem which may be proved on lines very similar to (*a*), (*b*), (*f*) above. It is that, if two triangles are self-polar with respect to a conic, their vertices lie on one conic and their sides touch another conic. Further, if one triangle can be constructed to have its vertices on a given conic and to be self-polar with respect to another conic, then an infinity of such triangles can be constructed, and there is a dual theorem to this.

23. Linear systems of conics.

(*a*) The conics

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

whose coefficients satisfy r linear equations

$$K_i \equiv \alpha_i a + \beta_i b + \gamma_i c + \theta_i f + \phi_i g + \psi_i h = 0,\ i = 1, \ldots, r,$$

are said to form a *linear system*

The r linear equations are said to be *dependent* if numbers $h_1, \ldots, h_r$ exist so that, for all a, b, c, f, g, h,

$$h_1K_1 + \ldots + h_rK_r = 0,$$

and otherwise to be *independent.* If these equations are independent and $r \leqslant 5$, we may solve them by Cramer's rule for any r of the coefficients which will then be expressed each as a linear homogeneous polynomial in the remaining $6 - r$ coefficients. On substituting these values in the equation first written, the system of conics is then expressed by an equation of the form

$$\lambda_0 S_0 + \lambda_1 S_1 + \ldots + \lambda_{5-r} S_{5-r} = 0,$$

where the λ_i are the remaining coefficients just mentioned and count as parameters which vary from conic to conic, and the S_i are definite independent homogeneous quadratic polynomials in x, y, z.

The system of conics, each of which depends on the ratios of $5-r$ of the λ_i to the remaining one, is called an ∞^{5-r} linear system or, alternatively, a linear system of *freedom* $5 - r$.

If $r = 5$, the system consists of a single conic. If $r = 4$ the system is called a *pencil*, and if $r = 3$ the system is called a *net.*

A geometrical condition on a system of conics which is expressible by a homogeneous linear equation in the coefficients $a, \ldots, h$ is called a *linear condition.* We have stated, then, that a conic is determined absolutely by five independent linear conditions.

(*b*) We prove that a single irreducible conic passes through five given points of which no three are in line.

The condition that conics should pass through a given point is linear. If then we can prove that the five given points present independent conditions we can conclude that a single conic contains all of them, and this conic must be irreducible since no pair of lines contains all five points.

Let us suppose that the five points present dependent conditions. Then we may solve the corresponding linear equations in $a, \ldots, h$ for not more than four of these coefficients; consequently the conics which pass through the given points are expressible by an equation of the form

$$\lambda_0 S_0 + \lambda_1 S_1 + \ldots + \lambda_t S_t = 0,$$

where the S_i are independent and $t \geqslant 1$. We may now choose the λ_i so that at least one of these conics contains a third point on the line joining two of the given points; such a conic would then contain the whole line and be reducible. As we have already said, no reducible conic contains the five given points; hence our supposition is impossible and the theorem follows.

(c) At least one conic passes through any five given points. If three of the points are in line and the other two lie on a different line, there is just one conic through the points; it consists of the two lines. If four of the points are in line, a pencil of conics contains the five points, each conic consists of the line mentioned and a line through the fifth point. If all five points are in line, a net of conics contains them; each conic consists of the line mentioned and another line.

(d) In order that the conic $S = 0$ should touch a given line $lx + my + nz = 0$ at a given point (x_1, y_1, z_1), the coefficients in S must satisfy two independent linear conditions.

By identifying the equation of the line with the equation $S_1 = 0$ we see that the two conditions are expressed by

$$ax_1 + hy_1 + gz_1 : hx_1 + by_1 + fz_1 : gx_1 + fy_1 + cz_1 = l : m : n,$$

these relations obviously containing the implication that $S_{11} = 0$ since $lx_1 + my_1 + nz_1 = 0$.

We may express the above statement by saying that the two infinitely near points in which the conic meets the line present two independent linear conditions to the coefficients in the equation of the conic. This could be expected as a limiting case of the conic being required to pass through two distinct given points on the given line.

It requires only slight modification of the argument given in (b) above to show that a single irreducible conic passes through four given points, of which no three are in line, and touches at one of them a given line not containing any of the other three. And, further, a single irreducible conic touches two given lines at given points and passes through a third given point, the three points not being in line nor any at the common point of the two lines.

Ex. 1. To require conics to have a given pair of points as conjugate points is a linear condition.

Ex. 2 To assign a self-polar triangle is equivalent to three independent linear conditions

Ex. 3. There is, in general, a unique conic having five given pairs of conjugate points of which no three pairs form the three pairs of opposite vertices of a quadrilateral.

Ex. 4. In general, two conics pass through four given points and touch a given line; but there is only one such conic if the line contains one of the given points.

Ex. 5. Assuming Pascal's theorem and (b) above, prove the converse of Pascal's theorem.

24. The intersections of two conics.

(*a*) To investigate the points of intersection of two conics A, B, at least one of which, say A, is irreducible, we choose a frame of reference so that A is given by the parametric equations

$$x : y : z = \lambda^2 : \lambda : 1.$$

Then, if B has the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

the points where B meets A have the parameters given by

$$a\lambda^4 + b\lambda^2 + c + 2f\lambda + 2g\lambda^2 + 2h\lambda^3 = 0.$$

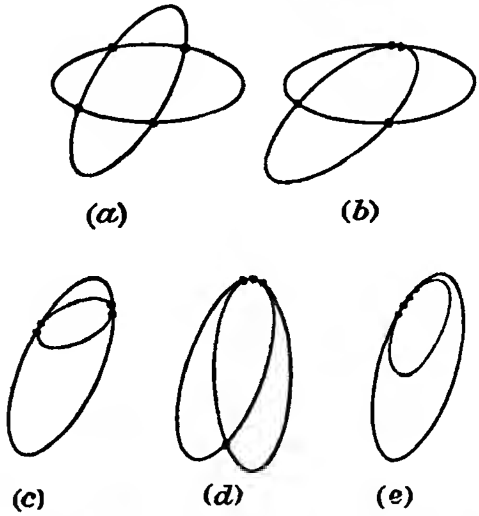


FIG 51.—INTERSECTIONS OF TWO CONICS, (*a*) DISTINCT INTERSECTIONS, (*b*) CONTACT AT ONE POINT, (*c*) DOUBLE CONTACT, (*d*) THREE-POINT CONTACT, (*e*) FOUR-POINT CONTACT

Therefore, in general, there are four distinct common points. If two of the roots of the quartic in λ are equal, we agree to say that B meets A twice at the corresponding point or that B has *contact of the first order* there with A; and similarly for contact of higher order (Fig. 51).

(*b*) To study the nature of the intersection at a particular common point P we may, without loss of generality, suppose that the frame of reference is so chosen that P has the parameter 0. Then necessary and sufficient conditions for B to (i) pass simply through P, (ii) meet A twice at P, (iii) meet A three times at P, (iv) meet A four times at P are respectively (i) $c = 0$, (ii) $c = f = 0$, (iii) $c = f = b + 2g = 0$, (iv) $c = f = b + 2g = h = 0$.

If P and Q are two different points of A, we can arrange that they have parameters 0, ∞ respectively. Then necessary and sufficient conditions for B to (v) meet A twice at P and twice at Q, (vi) meet A three times at P and once at Q are respectively (v) $c = f = a = h = 0$, (vi) $c = f = b + 2g = a = 0$.

If P, Q, R are three points on A, we can arrange that their parameters are 0, 1, ∞ respectively. Then necessary and sufficient conditions for B to (vii) meet A twice at P and once at each of Q, R are (vii) $c = f = a = b + 2g + 2h = 0$.

It is to be observed that in each case the conditions are linear and independent.

Ex. 1. If B has contact of the first or higher order with A at P, then the tangent at P to A is also tangent to B.

Ex. 2. It is very useful to recognise reducible conics B which satisfy the above conditions (i), (ii), . . . , or (vii). The following cases should be verified.

(i) Any line through P and any other line.

(ii) The tangent at P and any other line; or any pair of lines through P.

(iii) The tangent at P and any other line through P.

(iv) The tangent at P, counted twice.

(v) The tangents at P, Q, or the line PQ, counted twice.

(vi) The tangent at P and the line PQ.

(vii) The lines PQ, PR, or the tangent at P and the line QR.

25. Pencils of conics.

(i) A pencil is determined by any two of its members.—A system of conics given by an equation of the form

$$S + \lambda S' = 0,$$

where λ is a parameter varying from one conic to another and ranging through the modified system of complex numbers, and $S = 0$, $S' = 0$ are the equations of two different (reducible or irreducible) conics has been called a pencil.

The conics $S = 0$, $S' = 0$ may be said to determine the pencil. We prove that the pencil is equally well determined by any two of its members.

Consider then the two different conics $S + \lambda_1 S' = 0$, $S + \lambda_2 S' = 0$, where $\lambda_1 \neq \lambda_2$; these determine a pencil

$$(S + \lambda_1 S') + \mu(S + \lambda_2 S') = 0$$

or

$$(1 + \mu)S + (\lambda_1 + \mu\lambda_2)S' = 0.$$

The equation $\lambda = (\lambda_1 + \mu\lambda_2)/(1 + \mu)$ is bilinear in λ, μ and non-singular since $\lambda_1 \neq \lambda_2$. Therefore, to every value of μ there is one value of λ, and *vice versa*. Every member of either pencil is thus a member of the other pencil, that is, the two pencils are the same.

This property of a pencil is useful in building up the equation of a pencil, as we shall illustrate shortly.

It is evident that through any point, not common to $S = 0$, $S' = 0$, there passes just one conic of the pencil.

(ii) Base points of a pencil.—The common points of the conics of a pencil are also called the *base points* of the pencil. If the conics all touch a fixed line at P and have two other base points or else touch another line at a point Q, we say that there are two *infinitely near* base points on the line at P, and similarly at Q. If the conics all have contact of the second order with an irreducible member of the pencil at a point P, we say that there are three infinitely near base points on this member at P. And if the conics all have contact of the third order with an irreducible member at a point P, we say that there are four infinitely near base points on this member at P A base point is *isolated* if no other base point is infinitely near to it.

Underlying this terminology is the idea of the limit of two, three or four distinct base points on a curve It would not do to speak of three or four base points being infinitely near together on a line since such a case would require all the conics to be reducible and contain the line as part.

Ex. 1. Every conic through the base points of a pencil belongs to the pencil

Ex. 2. Suppose that the conic $S = 0$ is irreducible and show that, if the conic $S' = 0$ has contact of the rth order with $S = 0$ at a particular point, then every conic of the pencil $S + \lambda S' = 0$ has contact of the rth order there with $S = 0$.

Ex. 3. A is an isolated base point of the pencil $S + \lambda S' = 0$, and AP_i is the tangent at A to the conic $S + \lambda_i S' = 0$, $i = 1, 2, 3, 4$. Prove that $A\{P_1, P_2;\ P_3, P_4\} = \{\lambda_1, \lambda_2,\ \lambda_3, \lambda_4\}$.

Ex. 4. A, B are two isolated base points of a pencil of conics Prove that the tangents at A, B to a variable conic in the pencil correspond in a projectivity. What is the locus of the point in which corresponding tangents meet?

(iii) Special forms for the equation of a pencil.—Let the conics $S = 0$, $S' = 0$ be both irreducible and have four distinct common points. Then no three common points are in line. Let $L = 0$ be the equation of the line joining two of the common points and $M = 0$ be the line joining the other two; and let $L' = 0$, $M' = 0$ be another pair of opposite sides of this quadrangle of points. Then two reducible conics of the pencil $S + \lambda S' = 0$ are $LM = 0$ and $L'M' = 0$. The pencil is thus also given by either of the equations

$$S + \theta LM = 0,\ LM + \phi L'M' = 0$$

where θ, ϕ are parameters.

Let us suppose next that the conic $S' = 0$ touches $S = 0$ at P and meets it besides in two distinct points Q, R. Let $L = 0$ be the equation of the common tangent at P, $M = 0$ be QR, $L' = 0$ be PQ, $M' = 0$ be PR. Then two reducible conics of the pencil $S + \lambda S' = 0$ are $LM = 0$ and $L'M' = 0$. The pencil is thus given also by either of the equations

$$S + \theta LM = 0, \; LM + \phi L'M' = 0.$$

Next let $S' = 0$ meet $S = 0$ three times at P and once at a different point Q. Let $L = 0$ be the equation of the common tangent at P and $M = 0$ be PQ. The pencil $S + \lambda S' = 0$ is also given by

$$S + \theta LM = 0.$$

Now let $S' = 0$ meet $S = 0$ four times at P, the common tangent being $L = 0$. The pencil $S + \lambda S' = 0$ is given also by

$$S + \theta L^2 = 0.$$

Finally, let $S' = 0$ touch $S = 0$ at two different points P, Q, the common tangents there being $L = 0$, $M = 0$ respectively, and let PQ be $N = 0$. Then the pencil $S + \lambda S' = 0$ is given also by either of

$$S + \theta LM = 0, \; S + \phi N^2 = 0, \; LM + \psi N^2 = 0.$$

(iv) The reducible conics in a pencil.

Let $S = 0$, $S' = 0$, where

$$\begin{aligned} S &\equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy, \\ S' &\equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy, \end{aligned}$$

be the equations of two different conics These determine a pencil $S + \lambda S' = 0$ whose reducible members arise from the values of λ given by

$$\begin{vmatrix} a + \lambda a' & h + \lambda h' & g + \lambda g' \\ h + \lambda h' & b + \lambda b' & f + \lambda f' \\ g + \lambda g' & f + \lambda f' & c + \lambda c' \end{vmatrix} = 0,$$

that is, by

$$\Delta + \lambda\Theta + \lambda^2\Theta' + \lambda^3\Delta' = 0,$$

where Δ, Δ' have the usual significance and

$$\begin{aligned} \Theta &\equiv a'A + b'B + c'C + 2f'F + 2g'G + 2h'H, \\ \Theta' &\equiv aA' + bB' + cC' + 2fF' + 2gG' + 2hH', \end{aligned}$$

the capital letters denoting co-factors in Δ, Δ' as usual.

Thus, in general, a pencil contains three reducible conics. This occurs when the base points form a quadrangle, the reducible

conics consisting of the three pairs of opposite sides of the quadrangle.

In particular cases two or all three of the reducible conics may coincide; and if $\Delta = \Theta = \Theta' = \Delta' = 0$, every conic of the pencil is reducible. The reader is advised to consider the different pencils mentioned in part (iii) of this section and to find, by means of the cubic in λ, in each case the reducible members; for this purpose the equation $S = 0$ should be taken in its simplest form having regard to the configuration of base-points. Thus, for example, in the second case we should take PQR as triangle of reference and $S \equiv 2fyz + 2gzx + 2hxy$.

(v) The locus of the poles of a given line with respect to the conics of a pencil.—For simplicity, we consider only the case where the pencil has four distinct base-points, forming a quadrangle, and the given line is in general position with respect to the quadrangle

Let the base-points be A, B, C, D and let E, F, G be the diagonal points of the quadrangle With EFG as triangle of reference and the unit point suitably chosen, the equation of the pencil takes the form

$$(ax^2 + by^2 + cz^2) + \lambda(x^2 + y^2 + z^2) = 0.$$

The conic-envelope associated with the conic of parameter λ has the equation

$$\frac{l^2}{a + \lambda} + \frac{m^2}{b + \lambda} + \frac{n^2}{c + \lambda} = 0,$$

and the pole of the given line (l_1, m_1, n_1) with respect to this is given by

$$x : y : z = \frac{l_1}{a + \lambda} : \frac{m_1}{b + \lambda} : \frac{n_1}{c + \lambda}.$$

The locus of the pole is therefore the conic

$$\begin{vmatrix} ax & by & cz \\ x & y & z \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

or
$$\frac{l_1(b - c)}{x} + \frac{m_1(c - a)}{y} + \frac{n_1(a - b)}{z} = 0.$$

This conic passes through E, F, G, corresponding to the values $-a$, $-b$, $-c$ of the parameter It passes through the two points where the given line is touched by members of the pencil, corresponding to the values of the parameter given by

$$\frac{l_1^2}{a + \lambda} + \frac{m_1^2}{b + \lambda} + \frac{n_1^2}{c + \lambda} = 0.$$

There are six other points on the conic which deserve mention. Let the given line meet a side of the quadrangle, say AB, at X; and let $Y = (A, B)/X$. Then the conic contains the six points such as Y. To see this we observe that the polars of a point (x_1, y_1, z_1) form a pencil of lines given by $S_1 + \lambda S_1' = 0$. Thus the polars of Y form a pencil of lines, whose vertex is necessarily at X on account of the harmonic property of polar lines. Hence for just one conic of the pencil the polar of Y is the given line; that is, Y is the pole of the given line with respect to this conic.

The conic, which we have obtained as a locus of poles, is called the *eleven-point conic* of the quadrangle and line. We shall refer later to a metrical interpretation of the configuration (the nine-point circle of a triangle)

(vi) An involution property of a pencil.

(*a*) Let L be any line which does not contain a base point of the pencil of conics $S + \lambda S' = 0$. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be reference points on L, so that any point on L may be represented by a symbol $P_1 + \theta P_2$

The two points in which L meets the conic with parameter λ are given by

$$(S_{11} + 2\theta S_{12} + \theta^2 S_{22}) + \lambda(S_{11}' + 2\theta S_{12}' + \theta^2 S_{22}') = 0.$$

Therefore (by section 9 (vi) (e)), the two points are mates in an involution on L. This is a theorem of *Desargues* and *Sturm*

Two conics in the pencil touch L, the points of contact being the double points of the involution.

By considering the reducible members of the pencil, we have again the theorem that the pairs of opposite sides of a quadrangle meet a line in pairs of mates of an involution.

(*b*) If L were to pass through just one base point of the pencil, the above quadratic in θ would have one fixed root θ_0, corresponding to the base point, and one variable root, of the form $(p\lambda + q)/(r\lambda + s)$. In view of the bilinear relation thus existing between the parameter of the conic and the parameter of the variable intersection with L, we may say that the point and conic correspond projectively. In the present case, just one conic of the pencil touches L, its parameter being given by $p\lambda + q = \theta_0(r\lambda + s)$

Ex. 5. A variable conic through four points A, B, C, D meets a fixed conic through A, B at P, Q. Prove that PQ meets CD in a fixed point.

[Let the line CD meet the fixed conic at E, F and PQ at U, AB at V. The pencil of conics through A, B, P, Q determines an involution on CD in which C and D, E and F, U and V are pairs of mates. Therefore U is fixed.]

Ex. 6. Three conics through the points A, B meet again by pairs at C_1, D_1; C_2, D_2; C_3, D_3. Prove that C_1D_1, C_2D_2, C_3D_3 are concurrent.

[The conics through A, B, C_2, D_2 and through A, B, C_3, D_3 both pass through A, B, C_1, D_1; and C_2D_2, C_3D_3 are the lines joining their intersections, other than A, B, with the conic $ABC_2D_2C_3D_3$ The statement therefore follows from *Ex.* 5.

Algebraically, let $S = 0$ be the conic through A, B, C_2, D_2, C_3, D_3; let C_2D_2, C_3D_3, AB be respectively $L = 0$, $M = 0$, $N = 0$. Then the conic through A, B, C_1, D_1, C_2, D_2 has an equation of the form $S + \lambda LN = 0$; and the conic through A, B, C_1, D_1, C_3, D_3 has an equation of the form $S + \mu MN = 0$ A conic through the common points of these is $\lambda LN = \mu MN$, it contains AB as part; the other part is C_1D_1, whose equation is thus $\lambda L = \mu M$, and so is concurrent with C_2D_2, C_3D_3]

Ex 7. A, B, C, D are four variable points on a conic, such that AB, BC, CD pass respectively through three fixed collinear points P, Q, R Prove that DA meets PQR in a fixed point.

(vii) The line-equation of the base points.—The equation of the conic-envelope associated with the conic $S + \lambda S' = 0$ is

$$\Sigma + \lambda\Phi + \lambda^2\Sigma' = 0,$$

where

$$\begin{aligned}\Sigma &\equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm,\\ \Phi &\equiv (bc' + b'c - 2ff')l^2 + \dots\\ &\qquad\qquad + 2(gh' + g'h - af' - a'f)mn + \dots,\\ \Sigma' &\equiv A'l^2 + B'm^2 + C'n^2 + 2F'mn + 2G'nl + 2H'lm\end{aligned}$$

The conics in the pencil which touch a given line (l_1, m_1, n_1) therefore have parameters given by

$$\Sigma_{11} + \lambda\Phi_{11} + \lambda^2\Sigma_{11}' = 0.$$

As we have seen, in part (vi), there are two such conics, corresponding to the two roots of this quadratic in λ, when the given line does not contain a base point. When the given line contains a base point, there is only one conic touching it, and the quadratic in λ has then equal roots Thus the lines through the base points are those whose co-ordinates satisfy the equation

$$\Phi^2 = 4\Sigma\Sigma',$$

which is therefore the line-equation of the base points.

26. Ranges of conic-envelopes.

All that has been said in sections 23, 24, 25 may be dualised in a fairly obvious manner and gives rise to a set of statements

relating to conic-envelopes. We may be satisfied to rely on the algebra of the preceding sections and simply enumerate below the principal theorems in which we are interested.

(*a*) The system of conic-envelopes given by an equation of the form

$$\Sigma + \lambda\Sigma' = 0,$$

where λ is a parameter varying from one conic-envelope to another, is called a *range*.

Just as a pencil of conics is connected with the figure of a quadrangle, determined by its four base points, so a range of conic-envelopes is connected with the figure of a quadrilateral, determined by the four lines common to the members of the range.

In general, the conics associated with the conic-envelopes of a range do not form a pencil. With the notation of section 17, if $S = 0$, $S' = 0$ are the conics associated with $\Sigma = 0$, $\Sigma' = 0$ respectively, then the conic associated with $\Sigma + \lambda\Sigma' = 0$ has the equation

$$\Delta S + \lambda K + \lambda^2\Delta' S' = 0,$$

where the only new expression, K, is defined by

$$K \equiv (BC' + B'C - 2FF')x^2 + \dots \\ + 2(GH' + G'H - AF' - A'F)yz + \dots$$

Thus, the associated conics form a non-linear system of freedom 1.

(*b*) A range is determined by any two of its members.

(*c*) In general, a range contains three members which are pairs of pencils of lines. If every member contains the sides of a quadrilateral, the vertices of these pairs of pencils are the pairs of opposite vertices of the quadrilateral

The parameters of the reducible conic-envelopes are given by

$$\Delta^2 + \lambda\Delta\Theta' + \lambda^2\Delta'\Theta + \lambda^3\Delta'^2 = 0.$$

(*d*) The poles of a given line with respect to the conic-envelopes of a range lie on another line; and the polars of a given point belong to a conic-envelope.

(*e*) The pairs of lines of a variable conic-envelope in a range which pass through a fixed point are pairs of mates in an involution. This includes the theorem that the pairs of lines joining a given point to the pairs of opposite vertices of a quadrilateral are pairs of mates in an involution.

Of the associated conics of the range, two contain a given point which is not on a base line of the range; if the point is on a base line, then only one conic passes through it.

(*f*) The equation of the set of four base lines is

$$K^2 = 4\Delta\Delta' SS'.$$

(*g*) Let two irreducible conics have four distinct common tangents; no three may then be concurrent. The equation of the range determined by the associated conic-envelopes may be taken as $\Sigma + \lambda\Sigma' = 0$, where Σ, Σ' have obvious reference, or as $\Sigma + \mu LM = 0$ or as $LM + \nu L'M' = 0$, where $L = 0$, $M = 0$ are the line-equations of a pair of opposite vertices of the quadrilateral of common tangents, and $L' = 0$, $M' = 0$ similarly refer to another pair.

Suppose next that the two conics touch at a point where the common tangent is P and have two other distinct common tangents Q, R. Then the equation of the range may be taken as $\Sigma + \lambda\Sigma' = 0$ or $\Sigma + \mu LM = 0$ or $LM + \nu L'M' = 0$, where $L = 0$ is the equation of the common point of contact, $M = 0$ is the equation of Q R, $L' = 0$ is the equation of $\mathsf{P} . \mathsf{Q}$ and $M' = 0$ is the equation of P R

Now suppose that the two conics touch at points P, Q. Then the equation of the range may be taken as $\Sigma + \lambda\Sigma' = 0$ or $\Sigma + \mu LM = 0$ or $\Sigma + \nu N^2 = 0$ or $LM + \tau N^2 = 0$, where $L = 0$. $M = 0$, $N = 0$ are respectively the equations of P, Q and the common pole of PQ

If the two conics have three-point contact at P, then three of the four common tangents coincide with the tangent at P to both conics. In this case the range defined by the two conic-envelopes may be taken as $\Sigma + \lambda\Sigma' = 0$ or $\Sigma + \mu LM = 0$, where $L = 0$ is the line-equation of P and $M = 0$ is the equation of the point common to the tangent at P and the remaining common tangent

If the two conics have four-point contact at P, then the four common tangents coincide with the tangent at P to both conics. The range may be taken as $\Sigma + \lambda\Sigma' = 0$ or $\Sigma + \mu L^2 = 0$, where $L = 0$ is the equation of P.

Ex 1. If and only if the conics of a pencil have double or four point contact, the associated conic-envelopes form a range.

27. Rational curves and envelopes : projective transformations on conics and conic-envelopes.

(i) Rational curves and envelopes.

(*a*) A curve in the complex modified plane which has parametric equations of the form

$$x : y : z = A(\lambda) : B(\lambda) : C(\lambda),$$

where A, B, C are polynomials in a complex parameter λ, is called a *rational curve.*

Not every curve is rational: for example, it may be shown that the curve whose equation is expressed by the vanishing of a general cubic polynomial, homogeneous in x, y, z, is not rational; the co-ordinates of its points may be expressed as elliptic (that is, doubly-periodic) functions of a parameter.

The parametric equations, written above, define a single point (x, y, z) corresponding to every value of λ: but they do not necessarily define a single value of λ corresponding to every point on the curve. The proof is given later of a theorem due to Luroth which asserts that we can always find a *rational function* $f(\lambda)$ (that is, a function which is the ratio of two polynomials in λ) so that, on putting $f(\lambda) = \mu$, the parametric equations may be rewritten in the form

$$x : y : z = A_1(\mu) : B_1(\mu) : C_1(\mu),$$

where A_1, B_1, C_1 are polynomials in μ, and now every point on the curve, apart from a finite set of points, called *multiple points*, defines a single value of μ which can be expressed as a rational function of x, y, z.

For example, the conic $xz - y^2 = 0$ admits the parameterisation $x : y : z = \lambda^4 : \lambda^2 : 1$. Taking $f(\lambda) \equiv \lambda^2$, we get $x : y : z = \mu^2 : \mu : 1$, with $\mu = x/y = y/z$.

Dually, an algebraic system of lines given by parametric equations of the form

$$l : m : n = A(\lambda) : B(\lambda) : C(\lambda)$$

is called a *rational envelope*. It is evident that Luroth's theorem applies equally here.

(*b*) The simplest rational curves are:
the line

$$x : y : z = x_1 + \lambda x_2 : y_1 + \lambda y_2 : z_1 + \lambda z_2;$$

and the irreducible conic

$$x : y : z = a_1\lambda^2 + a_2\lambda + a_3 : b_1\lambda^2 + b_2\lambda + b_3 : c_1\lambda^2 + c_2\lambda + c_3,$$

with

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

The simplest rational envelopes are:
the pencil

$$l : m : n = l_1 + \lambda l_2 : m_1 + \lambda m_2 : n_1 + \lambda n_2;$$

and the irreducible conic-envelope

$$l : m : n = \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 : \beta_1\lambda^2 + \beta_2\lambda + \beta_3 : \gamma_1\lambda^2 + \gamma_2\lambda + \gamma_3,$$

with

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0.$$

(c) The algebra of bilinear equations, developed in section 9, may clearly be applied to rational curves and envelopes in the same way as to lines and pencils. Thus, a *projectivity on a conic* is determined by a set of parametric equations of the conic together with a non-singular bilinear equation connecting the parameters of two variable points on the conic.

(ii) Projectivity on a conic.

(a) We prove that the lines joining corresponding points in a non-involutory projectivity on an irreducible conic generate an irreducible conic-envelope whose associated conic touches the given conic at the united points of the projectivity. (See *Ex.* 2.)

If the conic is given by the general parametric equations in (i) (b) above, it is convenient first to change the triangle of reference by means of the substitution

$$x:y:z = a_1X + a_2Y + a_3Z \quad b_1X + b_2Y + b_3Z \,.\, c_1X + c_2Y + c_3Z,$$

so as to bring the equations to the simplest form

$$X : Y : Z = \lambda^2 : \lambda \,.\, 1.$$

Then let the projectivity be given by the bilinear equation

$$a\lambda\mu + b\lambda + c\mu + d = 0, \qquad \text{with } ad \neq bc,\ b \neq c.$$

A triad of new co-ordinates (l, m, n) of the chord joining corresponding points with parameters λ, μ is given by

$$l \,.\, m \quad n = 1 : -(\lambda + \mu) \cdot \lambda\mu.$$

The equation of the projectivity may be written as

$$a\lambda\mu + \tfrac{1}{2}(b + c)(\lambda + \mu) + d = -\tfrac{1}{2}(b - c)(\lambda - \mu);$$

from which

$$\{a\lambda\mu + \tfrac{1}{2}(b + c)(\lambda + \mu) + d\}^2 = \tfrac{1}{4}(b - c)^2\{(\lambda + \mu)^2 - 4\lambda\mu\}.$$

Therefore the chord in question generates the conic-envelope

$$\{an - \tfrac{1}{2}(b + c)m + dl\}^2 = \tfrac{1}{4}(b - c)^2\{m^2 - 4nl\}$$

The form of this equation shows that the associated conic touches the given conic at the points of contact of the tangents from the point $(d, -\tfrac{1}{2}(b + c), a)$. The polar of this point with respect to the given conic meets it in the points whose parameters are given by

$$d + a\lambda^2 + (b + c)\lambda = 0,$$

that is, in the united points of the projectivity.

(b) We prove that in the case of an involution on the conic the chords joining corresponding points pass through a fixed point, called the *centre of the involution*; and the double points are the points of contact of the tangents from this centre.

With the same preliminaries as in (*a*) above, let the equation of the involution be

$$a\lambda\mu + b(\lambda + \mu) + d = 0.$$

Then it is obvious that the chord joining a pair of mates passes through the point $(d, -b, a)$. The tangents from this point give rise to coincident mates, that is to the double points of the involution.

Ex. 1. $A_1A_2 \ldots A_n$ is a variable polygon inscribed in an irreducible conic, and $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ pass through fixed points, not on the conic. Show that, in general, A_1A_n touches a certain conic having double contact with the given conic; but that, if the fixed points are all in line, A_1A_n passes through a fixed point on their line, when n is even.

Show further that two polygons can be inscribed in a conic so as to have their sides each containing a given point.

Ex. 2. In the case of a parabolic projectivity on a conic, show that the conic-envelope generated by the lines joining corresponding points is such that its associated conic has four-point contact with the given conic at the united point of the projectivity.

Ex 3. A, B are two points conjugate with respect to an irreducible conic. P is any point on the conic, and AP, BP meet the conic again at R, S respectively. Prove that AS, BR meet on the conic.

Ex. 4. Two involutions on a conic are called *conjugate* when their centres are conjugate points. Let P, Q be mates in an involution and let P, R and Q, S be pairs of mates in a conjugate involution; then R, S are mates in the first involution.

(*c*) Problems relating to projectivities on a line may be very simply converted into problems relating to projectivities on a conic. To do this, we take any fixed point O on the conic and establish a projectivity between the line and the conic by making points correspond when they are in line with O. A projectivity on the line then gives rise to a projectivity on the conic.

The theorem, proved earlier in regard to involutions on a line, that two involutions have just one pair of mates in common is now seen to be an expression of the fact that the centres of two involutions on a conic are joined by one line which meets the conic in the common mates of the two involutions on the conic.

(*d*) It may be left to the reader to dualise the preceding notions and prove that the intersections of corresponding lines in a non-involutory projectivity on an irreducible conic-envelope generate a conic having double contact with the conic associated with the

conic-envelope; but, if the projectivity is involutory, the intersections lie on a fixed line.

(iii) The cross-axis of a projectivity.—We prove now that, if P_1 and Q_1, P_2 and Q_2 are two pairs of corresponding points in a projectivity on an irreducible conic, then the point of intersection of P_1Q_2 and P_2Q_1 lies on a fixed line, called the *cross-axis* of the projectivity.

When there are two distinct united points, it is evident, by taking P_1 at each in turn, that the cross-axis is the line joining the united points. If the projectivity is parabolic, the cross-axis must obviously pass through the one united point; it is in fact the tangent at the point since, if the line were to meet the conic elsewhere, this other intersection would necessarily be a united point; and this result is also apparent by considerations of continuity.

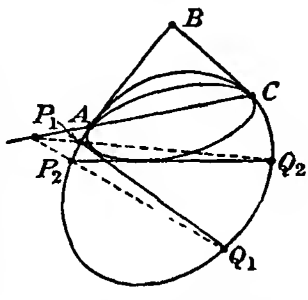


FIG 52.—CROSS-AXIS OF A PROJECTIVITY ON A CONIC.

The theorem may be deduced from Pascal's theorem in the same way that the existence of a cross-axis was deduced from Pappus' theorem in connection with a projectivity between two lines.

Algebraically, taking first the case of two distinct united points A, C we choose the triangle of reference to be ABC, where B is the pole of AC, and take the unit point on the conic (Fig. 52). The conic may then be given the parametric equations $x : y : z = \lambda^2 : \lambda : 1$, A, C having respectively the parameters ∞, 0, and consequently the projectivity is represented by an equation $\lambda = k\mu$.

If P_1, Q_1 have parameters λ_1, μ_1 and if P_2, Q_2 have parameters λ_2, μ_2 respectively, the equation of P_1Q_2 is

$$x - y(\lambda_1 + \mu_2) + z\lambda_1\mu_2 = 0.$$

This meets AC in the point $(\lambda_1\mu_2,\ 0,\ -1)$. Since $\lambda_1 = k\mu_1$, $\lambda_2 = k\mu_2$, these co-ordinates are the same as $(\lambda_2\mu_1,\ 0,\ -1)$, which refer to the point where P_2Q_1 meets AC. This is what we had to prove.

If the projectivity is parabolic, we take C to be the united point and A to be any other point on the conic, and again represent the conic by $x : y : z = \lambda^2 : \lambda : 1$. The equation of the projectivity now has the form $\lambda\mu = k(\lambda - \mu)$.

P_1, Q_1, P_2, Q_2 having the same parameters as before, P_1Q_2 meets the tangent at C in the point $(0,\ \lambda_1\mu_2,\ \lambda_1 + \mu_2)$. Since $\lambda_1 = k\mu_1/(k - \mu_1)$, these co-ordinates are the same as $(0,\ k\mu_1\mu_2,\ k(\mu_1 + \mu_2) - \mu_1\mu_2)$; and the result follows from the symmetrical form of these co-ordinates.

Ex. 5. In the case of an involution, the cross-axis is the polar line of the centre of the involution.

28. Review of Chapter III.

In this chapter we have investigated the basic properties of a conic and of its dual configuration, these being next in simplicity to a line and a pencil of lines respectively. The two types of geometrical assemblies proved to be closely associated: every irreducible conic-envelope is the assembly of tangents to an irreducible conic. It was shown that a conic and a conic-envelope can each be generated by purely linear geometrical constructions.

The reduction of the equation of a conic to various simple forms, depending on the relative positions of the triangle of reference, was, and frequently will be, very useful. In particular, one form led to the representation by means of parametric equations, showing that a conic belongs to the important class of curves called rational Much of the simplicity of the theory is a consequence firstly of this rationality and secondly of the simple polynomials in the parameter which are involved. And this rationality has been seen to widen the field of application of the theory of bilinear substitutions. Similar remarks apply to conic-envelopes.

A conic has been shown to possess a number of interesting harmonic properties Concepts of considerable use are those of conjugate points and lines and of pole and polar line.

Two distinct points serve to determine a line. Five points are required to determine a conic, and it is essential that they must present independent linear conditions; if the conic is to be irreducible, no three of the points may be in line.

Two lines meet in one point. Two conics meet in four points Both statements are particular cases of the important fundamental theorem that two algebraic curves of orders m and n meet in mn points Corresponding to the fact that an algebraic equation in one variable may have repeated roots, we introduce into geometry the notion of coincident, or infinitely close, intersections, for which limiting cases of distinct intersections provide a convenient mental picture.

The property of a pencil of conics, that it is determined by any two of its members, has wide application in forming the equation of a pencil. Frequently the most convenient members to choose are two out of the three reducible conics (line-pairs) which a general pencil has been shown to possess.

All the properties of conics and of conic-envelopes which have been dealt with so far are projective. That is to say, they are invariant under linear transformations of the co-ordinates. The theory of these transformations, which generalise to planes the

projective transformations on and between lines, is reserved for a later chapter.

The chapter has been written with regard to the geometry of a modified complex euclidean plane. It is clear that the same ideas apply to the other kinds of plane with which we are now familiar except for certain obvious modifications which should be everywhere apparent. Such a modification arises, for example, in the case of a modified real euclidean plane in respect of the fact that a quadratic equation need not have real roots.

CHAPTER IV

METRICAL THEORY OF CONICS

29. Preliminaries.

This chapter deals with metrical geometry, particularly that of conics. The subject is studied in a modified complex euclidean plane with reference to the various relations which may exist between geometrical configurations and the inaccessible line together with the circular points. Special attention is given to the metrical geometry of the real parts of the configurations.

Throughout the chapter, co-ordinates are modified rectangular distance co-ordinates. In any such system the equation of the inaccessible line is $z = 0$ and the circular points are I $(1, i, 0)$ and J $(1, -i, 0)$

The conic-envelope consisting of the pair of pencils of lines with vertices at I, J has the equation $(l + im)(l - im) = 0$, or $l^2 + m^2 = 0$. For brevity, we refer to this as the conic-envelope (I, J).

In accordance with conventions already made, the pole of a line (l_1, m_1, n_1) with respect to this conic-envelope is the harmonic conjugate with respect to I, J of the inaccessible point on the line and therefore has co-ordinates $(l_1, m_1, 0)$ Hence, the relation $l_1l_2 + m_1m_2 = 0$, necessary and sufficient that the lines (l_1, m_1, n_1), (l_2, m_2, n_2) should be perpendicular, is also an expression of the fact that the two lines should be conjugate with respect to the conic-envelope (I, J).

30. Metrical geometry of some linear configurations.

(1) The parallelogram.

(*a*) To conform with ordinary real euclidean geometry, we make the following obvious definitions

A *parallelogram* is a quadrilateral with two inaccessible opposite vertices; two sides which meet in such a vertex are parallel. The point of intersection of the accessible diagonals is called the *centre* of the parallelogram.

A *rectangle* is a parallelogram whose inaccessible vertices are harmonic with respect to I, J. The sides of one opposite pair are thus perpendicular to the other sides.

A *rhombus* is a parallelogram whose accessible diagonals are perpendicular. A *square* is a rectangular rhombus

(*b*) Let AB, CD be parallel sides of a parallelogram, the other two sides being BC and DA (Fig. 53). Let the corresponding inaccessible vertices be U, V. A line through the centre O meets AB, CD at P, Q respectively; we prove that O is the mid point of P, Q.

Let PQ meet UV at R By a harmonic property of the quadrangle $ABCD$, the lines UA, UD *harm* UO, UV. Hence, by incidence, P, Q *harm* O, R, since R is inaccessible, O is the mid point of P, Q

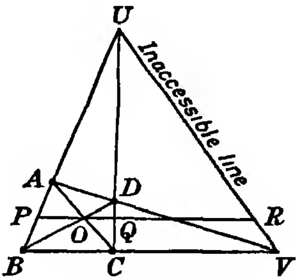


FIG. 53 —PARALLELOGRAM

Ex. 1. Prove that the angle from AB to BC is equal to the angle from DC to DA.

Ex 2. K, L, M, N are points on AB, BC, CD, DA respectively. Show that in general there are two rectangles $KLMN$.

Ex. 3 The equations of the sides of a parallelogram may be taken as $y = \pm bz$, $x = ay \pm cz$. Show that the accessible diagonals are $bx = (ab \pm c)y$. The parallelogram is a rectangle when $a = 0$; a rhombus when $b^2(1 + a^2) = c^2$, a square when $a = 0$, $b = \pm c$.

The following exercises are based on Pappus' theorem.

Ex 4. O, A, B, C are the accessible vertices of a parallelogram, with O opposite to B, and O, A', B', C' are those of another, with A' on OA, C' on OC Prove that AC', $A'C$ intersect on BB'.

Ex. 5 A, B, C are three points in line and P, Q, R are three other points such that AQ, BR, BP, CQ; CR, AP are three pairs of parallel lines. Prove that P, Q, R are in line

Ex. 6 P is a point on the side BC of a triangle ABC. Any line through A meets the parallels through P to AB, AC in R, S respectively. Prove that BS is parallel to CR

Ex. 7 A, B, C are points on one line and A', B', C' are points on another line, such that AA', CC' and AB', BC' are two pairs of parallel lines Prove that BA', CB' are parallel.

The following exercise is based on Desargues' theorem for triangles in perspective

Ex 8 A, A', B, B', C, C' are three pairs of points, each pair being in line with another point O. If AB, $A'B'$ are parallel and AC, $A'C'$ are parallel, then BC, $B'C'$ are parallel.

(ii) Properties of a triangle.—The figure of harmonic pole and polar line with respect to a triangle has special interest. The triangle being ABC the pole being O, AO meets BC in A', and so

on, and $B'C'$ meets BC in A'', and so on. A'', B'', C'' are on the polar line of O. The figure may equally well be constructed by starting with the polar line.

If we take $A''B''C''$ to be the inaccessible line, A' becomes the mid point of B, C since A', A'' *harm* B, C. Thus it follows that

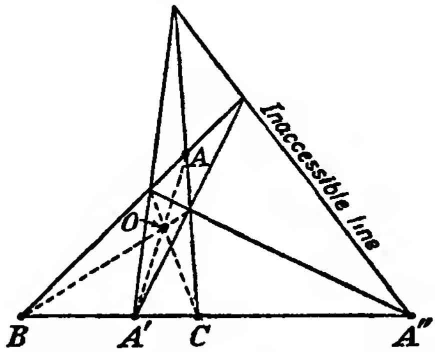


FIG. 54 —MEDIANS OF A TRIANGLE ABC.

the medians of a triangle are concurrent, in the pole of the inaccessible line; and that the line joining the mid points of two sides is parallel to the third side of the triangle (Fig. 54).

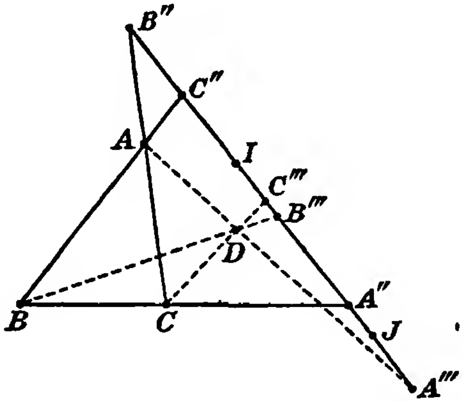


FIG 55 —ORTHOCENTRE OF A TRIANGLE ABC.

Now let A''' be the harmonic conjugate of A'' with respect to the circular points I, J, and so on (Fig 55) Then AA''' is perpendicular to BC, and so on. Let BB''', CC''' meet at D. The pairs of opposite sides of quadrangle $ABCD$ meet IJ in pairs of mates in an involution (see page 83), with I, J as double points, therefore CD contains C'''. Hence, the three perpendiculars

from the vertices to the opposite sides of a triangle are concurrent, in a point called the *orthocentre* of the triangle. And, in fact, each of the points A, B, C, D is the orthocentre of the triangle formed by the other three.

31. Metrical geometry of conics : generalities.

(1) Definitions.—Besides investigating the different possible relations which may exist between a conic and the inaccessible line and the circular points, we show how to recognise these relations from the form of the equation of the conic.

Let the equation of the conic be

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The conic is called a *parabola* when it is irreducible and touches the inaccessible line (Fig 56). In the usual notation, a parabola is characterised by $\Delta \neq 0$, $C = 0$ It is convenient to call a pair of parallel lines or a line counted twice an *improper parabola* since

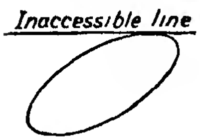


FIG. 56 —PARABOLA

FIG. 57 —CIRCLE.

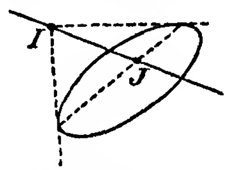


FIG 58.—RECTANGULAR CONIC.

such a reducible conic has two coincident intersections with the inaccessible line.

The conic is called a *circle*, as we have already agreed, when it contains the circular points (Fig. 57). If it is irreducible we call it a *proper circle*, if it reduces to two accessible lines we call it a *point-circle*, and if it reduces to the inaccessible line together with any other line, we call it an *improper circle*. A proper circle is characterised by $\Delta \neq 0$, $a = b$, $h = 0$, a point-circle by $\Delta = 0$, $a = b$, $h = 0$, an improper circle by $\Delta = a = b = h = 0$.

The conic is called *rectangular* when it meets the inaccessible line in points which are harmonic with respect to the circular points (Fig 58). For this, it is necessary and sufficient that the pair of lines $ax^2 + 2hxy + by^2 = 0$ should be harmonic with respect to the pair $x^2 + y^2 = 0$; and this happens if and only if $a + b = 0$. A reducible rectangular conic is a pair of perpendicular lines or an isotropic line counted twice.

Ex. 1. There are two parabolas in a pencil of conics having all its base points accessible.

Ex. 2. If two conics in a pencil are circles, then all the conics are circles.

Ex. 3. In a pencil of conics there is, in general, just one rectangular conic, if there are two, then every conic in the pencil is rectangular and they meet the inaccessible line in the pairs of mates of the involution having I, J as double points.

Ex. 4. Deduce from the second part of *Ex.* 3 that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

Ex. 5 The conics through the vertices and orthocentre of a triangle form a pencil of rectangular conics.

Ex. 6. The rectangular conics through the vertices of a triangle pass also through the orthocentre and form a pencil.

(ii) Centre of a conic.

(*a*) An irreducible conic, which is not a parabola, is called *central.* Its *centre* is defined to be the pole of the inaccessible line (Fig 59).

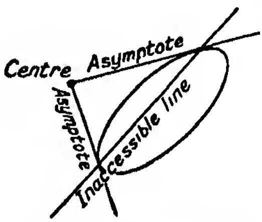


FIG 59 —CENTRE AND ASYMPTOTES OF A CENTRAL CONIC.

If a line through the centre O meets the conic at P, Q and the inaccessible line at R, then O, R *harm* P, Q; and therefore O is the mid point of P, Q This property characterises the centre

The equation of the conic being $S = 0$, the centre, as the pole of $z = 0$, is determined by the equations

$$ax + hy + gz = 0, \; hx + by + fz = 0$$

and therefore has co-ordinates (G, F, C)

The tangents from the centre are called the *asymptotes* of the conic. These are evidently perpendicular if and only if the conic is rectangular.

The asymptotes form a reducible conic belonging to the pencil determined by the conic and the inaccessible line counted twice. Their equation is therefore of the form

$$S + \lambda z^2 = 0$$

subject to $\lambda \neq \infty$ and

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+\lambda \end{vmatrix} = 0.$$

From the determinantal equation, $C\lambda + \Delta = 0$ The asymptotes are therefore given by

$$CS = \Delta z^2.$$

(*b*) A reducible conic is also said to have a centre in the following cases. A centre O is defined in each case so that if P

is on the conic and if O is the mid point of P, Q, then Q is on the conic.

If the conic consists of two non-parallel accessible lines, the centre is the common point of the two lines.

If the conic consists of two parallel accessible lines, any accessible point on their mid line (that is, the harmonic conjugate of the inaccessible line) is a centre.

If the conic consists of an accessible line counted twice, any accessible point on the line is a centre.

Ex. 7. Obtain the equation of the asymptotes of a central conic by writing down the equation of the pair of tangents from the point (G, F, C)

Ex. 8. In each of the reducible cases in (*b*) above, if the equation of the conic is $S = 0$, the centre or line of centres is given by

$$ax + hy + gz = 0, \; hx + by + fz = 0,$$

the two equations representing the same line, or one being satisfied identically, if there is a line of centres.

(iii) Diameters of a conic.

(*a*) A line passing through the centre of a central conic is called a *diameter*. Two diameters which are conjugate lines are a pair of mates in the involution which has the asymptotes as double lines (Fig 60).

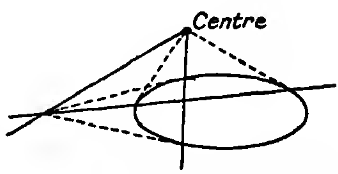


FIG. 60.—CONJUGATE DIAMETERS OF A CENTRAL CONIC.

The lines which pass through the pole of a diameter are said to have the *direction conjugate to that of the diameter*. If such a line meets the conic at P, Q and the diameter at R, then R is the mid point of P, Q.

The polar of the point $(1, \lambda, 0)$ with respect to the conic is the diameter

$$(ax + hy + gz) + \lambda(hx + by + fz) = 0,$$

and this meets the inaccessible line in the point $(1, \mu, 0)$ where

$$a + h(\lambda + \mu) + b\lambda\mu = 0.$$

Here, λ and μ may be regarded as parameters of conjugate diameters. The parameters of the asymptotes are given by

$$a + 2h\lambda + b\lambda^2 = 0;$$

therefore the equation of the asymptotes may also be expressed in the form

$$a(hx + by + fz)^2 - 2h(hx + by + fz)(ax + hy + gz) \\ + b(ax + hy + gz)^2 = 0.$$

(*b*) In the case of a parabola, a *diameter* is defined to be any accessible line through the point of contact, now $(G, F, 0)$, of the inaccessible line. Thus all diameters are parallel and conjugate to the inaccessible line

The pole of a diameter is inaccessible, taking it to have co-ordinates $(1, \lambda, 0)$ and putting $\theta = a/h = h/b$, the equation of the diameter is

$$b(\theta + \lambda)(\theta x + y) + (g + \lambda f)z = 0.$$

The vertex of the parallel pencil of diameters—that is, the point of contact of the inaccessible line, has thus the alternative triad of co-ordinates $(1, -\theta, 0)$.

Ex. 9. The tangents at the points where two conjugate diameters of a central conic meet the conic form a parallelogram.

Ex 10. P is an accessible point on a central conic and the tangent at P meets the asymptotes at A, B. Prove that P is the mid point of A, B.

A line parallel to this tangent meets the conic at Q, R and the asymptotes at C, D Prove that Q, R and C, D have the same mid point

(iv) Axes of a conic.—A diameter is said to be a *principal diameter* or *axis* of the conic when it is perpendicular to the lines having the conjugate direction

(*a*) In the case of a central conic, the diameter conjugate to an axis is also an axis. The axes are the pairs of mates common to the involution of pairs of conjugate diameters and to the involution of pairs of perpendicular lines (*orthogonal involution*) through the centre. If the conic is a proper circle these involutions are identical, and every diameter is an axis' Apart from this case, the involutions have just one pair of mates in common. Thus, in general, a central conic has just two axes, these being perpendicular to each other.

Algebraically, the parameters of the axes are given (*cf* (iii) (*a*)) by

$$a + h(\lambda + \mu) + b\lambda\mu = 0,$$
$$1 + \lambda\mu = 0,$$

and are therefore the roots of the quadratic in t

$$h(t^2 - 1) + (a - b)t = 0.$$

The equation of the pair of axes is therefore

$$h\{(ax + hy + gz)^2 - (hx + by + fz)^2\}$$
$$= (a - b)(ax + hy + gz)(hx + by + fz)$$

or

$$\frac{\left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2}{a - b} = \frac{\left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial S}{\partial y}\right)}{h}.$$

This equation may also be derived as follows. A point (x_1, y_1, z_1) on an axis is characterised by the fact that its polar line is perpendicular to the diameter through the point. The polar line is given by

$$x\left(\frac{\partial S}{\partial x}\right)_1 + y\left(\frac{\partial S}{\partial y}\right)_1 + z\left(\frac{\partial S}{\partial z}\right)_1 = 0$$

and the diameter by

$$\left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial S}{\partial y}\right)_1 = \left(\frac{\partial S}{\partial y}\right)\left(\frac{\partial S}{\partial x}\right)_1.$$

For these lines to be perpendicular it is necessary and sufficient that

$$\left(\frac{\partial S}{\partial x}\right)_1\left\{a\left(\frac{\partial S}{\partial y}\right)_1 - h\left(\frac{\partial S}{\partial x}\right)_1\right\} + \left(\frac{\partial S}{\partial y}\right)_1\left\{h\left(\frac{\partial S}{\partial y}\right)_1 - b\left(\frac{\partial S}{\partial x}\right)_1\right\} = 0,$$

that is, that the co-ordinates (x_1, y_1, z_1) should satisfy the equation already found.

(*b*) A parabola has only one axis, namely the polar of the point conjugate, with respect to the circular points, of the point of contact of the inaccessible line.

The point of contact having co-ordinates $(1, -\theta, 0)$, its conjugate with respect to I, J has co-ordinates $(\theta, 1, 0)$. The equation of the axis is therefore (*cf* (iii) (*b*))

$$\theta(ax + hy + gz) + (hx + by + fz) = 0,$$

or

$$b(\theta^2 + 1)(\theta x + y) + (g\theta + f)z = 0.$$

(v) Foci of a conic.

(*a*) The tangents from the circular points to a central conic form a quadrilateral of isotropic lines whose accessible vertices are called *foci* of the conic (Fig. 61); opposite foci are also called *associated* or *corresponding* foci.

A focus is characterised by being the vertex of an orthogonal involution of conjugate lines. The diameter through a focus is therefore an axis, the axes are in fact the lines joining the pairs

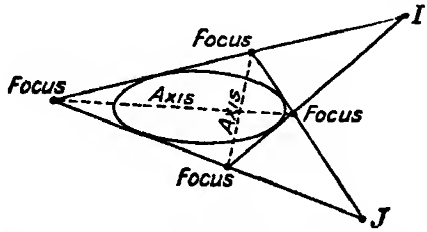


FIG. 61 —FOCI AND AXES OF A CENTRAL CONIC.

of opposite foci. A focus is also characterised by the fact that the tangents from the point constitute a point-circle.

Using the second characterisation, we observe that the tangents from the point (x_1, y_1, z_1) are given by

$$SS_{11} = S_1^2$$

and that this equation represents a point-circle if and only if

$$aS_{11} - (ax_1 + hy_1 + gz_1)^2 = bS_{11} - (hx_1 + by_1 + fz_1)^2,$$
$$hS_{11} = (ax_1 + hy_1 + gz_1)(hx_1 + by_1 + fz_1).$$

The foci may therefore be determined as the intersections of two conics whose equations may conveniently be combined in the form

$$\frac{\left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2}{a - b} = 4S = \frac{\left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial S}{\partial y}\right)}{h}.$$

(*b*) Through each circular point one accessible tangent can be drawn to a parabola. The point of intersection of these tangents is the *focus* of the parabola (Fig 62).

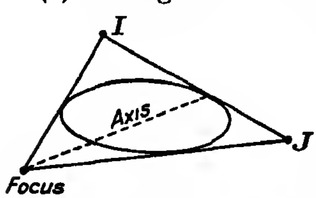


FIG. 62 —FOCUS AND AXIS OF A PARABOLA

The focus of a parabola is characterised in the same way as a focus of a central conic The axis of the parabola joins the focus to the point of contact of the inaccessible line.

(*c*) We consider the conic-envelope

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

associated with an irreducible conic $S = 0$.

If (x, y, z) is a focus, the co-ordinates of the tangents from the point satisfy the equations $\Sigma = 0$ and

$$lx + my + nz = 0,$$

and therefore also the equation, obtained by eliminating n,

$$(Al^2 + 2Hlm + Bm^2)z^2 + (2Fm + 2Gl)(-lx - my)z + C(-lx - my)^2 = 0.$$

This quadratic in l/m determines the two tangents; the roots of the quadratic have also to be such as to satisfy the equation

$$l^2 + m^2 = 0.$$

Therefore

$$Az^2 - 2Gzx + Cx^2 = Bz^2 - 2Fyz + Cy^2,$$

and

$$Hz^2 - Gyz - Fzx + Cxy = 0.$$

Thus, the foci are at the intersections of the two rectangular conics

$$C(x^2 - y^2) - 2(Gx - Fy)z + (A - B)z^2 = 0,$$
$$Cxy - (Fx + Gy)z + Hz^2 = 0.$$

In the case of a parabola, $C = 0$, and these conics are reducible The focus, being accessible, is determined by the accessible components

$$-2(Gx - Fy) + (A - B)z = 0,$$
$$-(Fx + Gy) + Hz = 0.$$

(vi) The director locus.—A locus of special interest, which we generalise later, is that of the accessible points from which can be drawn a pair of perpendicular tangents to an irreducible conic $S = 0$. It is called the *director* or *orthoptic locus*

If (x, y, z) is such a point and if (l_1, m_1, n_1), (l_2, m_2, n_2) are the tangents from it, l_1/m_1 and l_2/m_2 are the roots of the quadratic in l/m.

$$(Al^2 + 2Hlm + Bm^2)z^2 + 2(Fm + Gl)(-lx - my) + C(-lx - my)^2 = 0.$$

The tangents are perpendicular if and only if $l_1l_2 + m_1m_2 = 0$, that is, if and only if

$$C(x^2 + y^2) - 2(Gx + Fy)z + (A + B)z^2 = 0.$$

In the case of a central conic, the locus is a concentric circle, called the *director circle* of the conic (Fig. 63). In the case of a parabola ($C = 0$), the locus is the accessible line

$$2Gx + 2Fy = A + B,$$

called the *director line* of the parabola.

The polar of a focus is called a *directrix*. At a point common to the conic and a directrix, the two tangents from the point coincide with the self-perpendicular isotropic line which is the tangent at the point. Thus the director circle of a central conic passes through the four points where the directrices meet the conic, this property is also seen, perhaps more readily, by observing that the director-circle is the locus of the intersections of conjugate lines through I, J. And, in the case of a parabola, we see that the director line is the single directrix of the parabola.

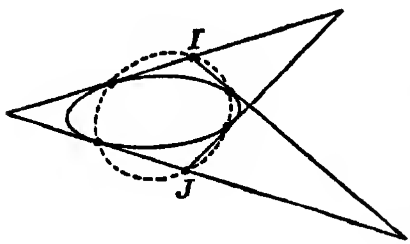


FIG. 63.—DIRECTOR CIRCLE OF A CENTRAL CONIC

Ex 11. The lines joining the accessible point (x, y, z) to I, J in turn have co-ordinates $(iz, -z, y - ix)$, $(-iz, -z, y + ix)$ These are conjugate with respect to $\Sigma = 0$ if and only if

$$Az^2 + Bz^2 + C(y^2 + x^2) + 2F(-yz) + 2G(-zx) = 0,$$

that is, if and only if the point lies on the director locus

Ex. 12 The tangents from a point (x_1, y_1, z_1) to $S = 0$ are $SS_{11} = S_1^2$. These meet the line $z = 0$ in the same points as do the lines

$$(ax^2 + 2hxy + by^2)S_{11} = \{x(ax_1 + hy_1 + gz_1) + y(hx_1 + by_1 + fz_1)\}^2$$

This pair of lines is harmonic with respect to the pair of lines $x^2 + y^2 = 0$ if and only if the point lies on the conic

$$(a + b)S = (ax + hy + gz)^2 + (hx + by + fz)^2.$$

Verify that this equation can be rearranged in the form given in the text.

(vii) The nine-point circle of a triangle.—A locus which deserves notice is that of the centres of the rectangular conics which pass through the vertices of a triangle and necessarily through the orthocentre as well. This is the eleven-point conic associated with the inaccessible line and therefore passes through (i) the feet of the perpendiculars from the vertices to the opposite sides, (ii) the mid points of the sides, (iii) the mid point of each vertex and the orthocentre. The rectangular conics which touch the inaccessible line must do so at the circular points Hence, there is

a circle through the nine points enumerated on the triangle; it is called the *nine-point circle* of the triangle (Fig. 64).

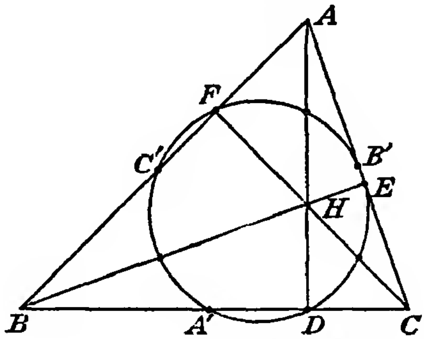


FIG. 64.—NINE-POINT CIRCLE OF A REAL TRIANGLE.

(viii) Normals to a conic.—The polar P of a point P (x_1, y_1, z_1) with respect to the irreducible conic $S = 0$ has the equation

$$x(ax_1 + hy_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0$$

It meets the inaccessible line in the point $(hx_1 + by_1 + fz_1, -ax_1 - hy_1 - gz_1, 0)$, whose harmonic conjugate with respect to the circular points is $(ax_1 + hy_1 + gz_1, hx_1 + by_1 + fz_1, 0)$. This second point is joined to P by the line Q, perpendicular to P, whose equation is

$$(x_1z - xz_1)(hx_1 + by_1 + fz_1) = (y_1z - yz_1)(ax_1 + hy_1 + gz_1).$$

When P is accessible and on the conic, Q is called the *normal* at P, and P is called the *foot* of the normal (Fig. 65).

The line Q passes through a fixed point Q (x_0, y_0, z_0) if and only if P lies on the conic

$$(xz_0 - x_0z)(hx + by + fz) = (yz_0 - y_0z)(ax + hy + gz).$$

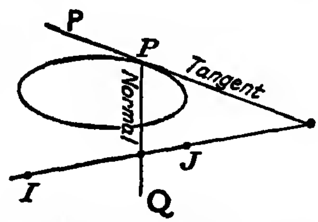


FIG. 65.—TANGENT AND NORMAL AT A POINT TO A CONIC.

This conic is, then, the locus of points P such that PQ is perpendicular to the polar of P. It is a rectangular conic passing through Q.

If the given conic is central and not a circle, it is obvious geometrically that two positions for P are the inaccessible points on the axes. The asymptotes of the rectangular conic are therefore parallel to the axes of the given conic. Further, the rectangular conic passes through the centre of the given conic.

If the given conic is a proper circle, the rectangular conic is reducible and composed of the inaccessible line together with the line joining Q to the centre of the circle.

If the given conic is a parabola, one position for P is the point of contact of the inaccessible line and another is the inaccessible point on the tangent at the *vertex*, the vertex being the accessible intersection of the axis with the parabola. Thus the asymptotes of the rectangular conic are parallel to the axis and tangent at the vertex of the parabola.

The rectangular conic meets the given conic in four points which are, in general, all accessible; but in the case of a parabola only three are accessible, the fourth being the point of contact of the inaccessible line, and in the case of a proper circle only two are accessible and are in line with Q and the centre. Hence, from a given point, four normals can be drawn to a central conic, three to a parabola, and two (coincident) to a circle. The feet of these normals are called *co-normal* points.

Ex. 13. P is a fixed point and Q, R are variable points on an irreducible conic, and PQ is perpendicular to PR. Prove that QR passes through a fixed point, called the *Frégier point* of P, on the normal at P.

32. Real conics.

We are interested here in the real points which belong to a conic—that is, in the part of the conic lying in the real plane which is embedded in the complex plane. Since a reducible conic consists of a pair of lines, which may coincide, and about which there is little really new to say, we confine attention to irreducible conics.

(i) Preliminaries.

(*a*) Let us suppose that all the coefficients in the equation of a given conic are real. It is obvious, by consideration of the particular equation $x^2 + y^2 + z^2 = 0$, that such a conic need not possess any real points. On the other hand, if the conic has at least one real point, it has an infinity of them since every real line through the point meets the conic again in another real point; and the co-ordinates of this residual intersection are proportional to real quadratic functions of the real parameter of the variable line. Thus a conic of the second type has a continuous curve of points in the real part of the plane, it is called a *real conic*.

Next, let us suppose that the equation contains complex coefficients. It may then be written in the form $S' + iS'' = 0$, where the coefficients in S', S'' are real. The real points are then common to the two conics $S' = 0$, $S'' = 0$, and are therefore 0,

2 or 4 in number, unless the two polynomials S', S'' are proportional or $S' \equiv 0$, in which case we are back to the circumstances of the previous case after dividing through the given equation by one complex number.

Ex. 1. The conic $(x^2 + y^2 - 8z^2) + i(xy - k^2z^2)$, where k is real, has four real distinct points when $0 \leqslant k < 2$; two distinct pairs of coincident real points when $k = 2$; and no real points when $k > 2$.

[Any real points are common to the conics $x^2 + y^2 - 8z^2 = 0$, $xy - k^2z^2 = 0$.]

(*b*) In the rest of this section, a set of co-ordinates referring to a real point will be any triad of real numbers appropriate to the point.

(ii) Interior and exterior points of a real conic.

(*a*) The real points on the real line joining the real points P_1 (x_1, y_1, z_1), P_2 (x_2, y_2, z_2) have co-ordinates of the form $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2)$, with λ real. Those which lie on the real conic $S = 0$, all the coefficients in S being real, have their parameters given by

$$S_{11} + 2\lambda S_{12} + \lambda^2 S_{22} = 0.$$

There are two distinct real points common to the line and conic if $S_{12}^2 > S_{11}S_{22}$, two coincident real intersections if $S_{12}^2 = S_{11}S_{22}$, and no real intersection if $S_{12}^2 < S_{11}S_{22}$. This property may be used to separate the points of the real plane into three classes relative to the real conic.

An *interior point* is a real point whose polar line, which is also real, has no real intersection with the conic, an *exterior point* is a real point whose polar meets the conic in two distinct real points (Fig. 66). The remaining points, whose polars touch the conic, are the points of the conic itself.

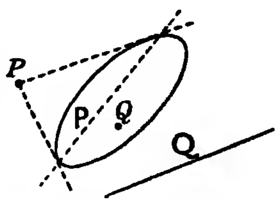


FIG. 66.—EXTERIOR AND INTERIOR POINTS.

(*b*) Now let P_1 (x_1, y_1, z_1), P_2 (x_2, y_2, z_2) be any two exterior points; and let a tangent from P_1 meet a tangent from P_2 at P_3 (x_3, y_3, z_3); since the tangents are real, so is P_3.

Since P_3 is on a tangent from P_1, and *vice versa*, we have $S_{11}S_{33} = S_{13}^2 > 0$; and similarly $S_{22}S_{33} > 0$. Hence S_{11}, S_{22} (and S_{33}) have the same sign Therefore the sign of S, evaluated at any exterior point, is the same; and, conversely, if P_4 is any real point such that $S_{11}S_{44} > 0$, then P_4 is an exterior point.

Since S vanishes only at the points of the conic, it follows that S has the opposite sign when evaluated at any interior point. We thus have an algebraic means of distinguishing between interior and exterior points.

(*c*) Consider now a real line meeting the real conic $S = 0$ in the real distinct points P_1 (x_1, y_1, z_1), P_2 (x_2, y_2, z_2). Then $S_{11} = S_{22} = 0$, $S_{12} \neq 0$. Any real point on the line P_1P_2 has co-ordinates of the form $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2)$, with λ real; and the value of S at this point is $2\lambda S_{12}$. The harmonic conjugate of this point with respect to P_1, P_2 has the parameter $-\lambda$; and S has the value $-2\lambda S_{12}$ there. Hence, if one of the points is interior, the other is exterior, and *vice versa*.

The real points on the line, but not on the conic, fall into two continua, those for which $\lambda > 0$ and those for which $\lambda < 0$. That continuum which consists of interior points is called the *interior chord* P_1P_2, and the other the *exterior chord* P_1P_2.

(*d*) We now prove that every real line through an interior point meets the conic in two real points.

The polar of the point, P, has no real intersections with the conic and so its real points are all interior or all exterior. Suppose that they are interior points and let Q be any one. Every real point on the polar of Q is interior, since P is one of them. As Q varies on the polar of P, the polar of Q varies in the pencil with vertex at P and describes the whole real plane. Therefore every point of the plane is an interior point. This conclusion is false; so therefore is the assumption that the points on the polar of P are interior, therefore they are exterior.

Any real line through P is the polar of a real point Q on the polar of P, since Q is exterior, the line through P meets the conic in two real points. (This conclusion may also be reached by considering the changes in sign as S varies continuously along the line through P.)

On the other hand, the polar of an exterior point meets the conic in two real points at which $S = 0$. The points on the polar, not at the intersections, fall into two sets, interior and exterior. Every line joining the original point to an interior point on the polar has two real intersections with the conic; every line joining the original point to an exterior point on the polar is the polar line of one of the interior points, and therefore all its points are exterior, so the line does not meet the conic in any real point.

Ex. 2. A conic having five real points is a real conic.

Ex 3. Determine the set of interior points of the conic $x^2 - 3y^2 = z^2$.

(iii) Ellipse, hyperbola and real parabola.—A real conic which has distinct real intersections with the inaccessible line is called

a *hyperbola*. Its centre and asymptotes are real; so are the axes. The centre is an exterior point.

A real conic which has no real intersections with the inaccessible line is called an *ellipse*. Its centre and axes are real but the asymptotes are not. The centre is an interior point.

Taking as triangle of reference that formed by the axes and the inaccessible line, which we retain as $z = 0$, the equation of a central conic takes the form

$$ax^2 + by^2 = z^2.$$

A hyperbola is characterised by a, b real with $ab < 0$; an ellipse by a, b real and positive.

In the case of a rectangular hyperbola it is often convenient to use a different triangle of reference, namely, the asymptotes, taken as $x = 0$, $y = 0$, and the inaccessible line, $z = 0$. The equation then has the form

$$xy = kz^2$$

with k real.

In the case of a real parabola, it is usually convenient to take the axis, which is real, as $y = 0$ and the tangent at the vertex, which is real and perpendicular to the axis, as $x = 0$, the inaccessible line being $z = 0$ The equation then has the form

$$y^2 = 4azx$$

with a real. The factor 4 is inserted because the number a is capable of a simple geometrical interpretation.

In the next three sections we discuss these equations in detail

33. The ellipse.

(i) The standard form of the equation.—In this section we are concerned with the properties of an ellipse in the accessible real part of the plane. All numbers are therefore to be taken as real and co-ordinates are ordinary rectangular distance co-ordinates. It should perhaps be emphasised that only a partial picture of the ellipse is thus presented; the background of the containing modified complex plane should be kept in mind.

It is usual and convenient to take the equation of the ellipse, referred to its axes, in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where we may suppose without loss of generality that $a > b > 0$.

(The reader is assumed to be familiar with the case of a proper circle, arising when $a = b$.)

(ii) Elementary properties.—The centre is an interior point, as we have already remarked; hence the set of interior points

consists of those for which

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}<1;$$

and the set of exterior points consists of those for which

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}>1.$$

The property of the centre, that it is the mid point of all interior diameters, appears simply in the fact that if the point (x, y) is on the conic so is the point $(-x, -y)$. Moreover, since also the point $(-x, y)$ is then on the ellipse, the conic is symmetrical about the y-axis; and, similarly, it is symmetrical about the x-axis (Fig. 67).

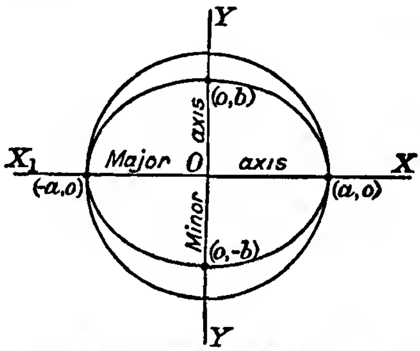


FIG 67.—THE ELLIPSE AND ITS AUXILIARY CIRCLE.

The ellipse meets the x-axis in the points $(\pm a, 0)$ and the y-axis in the points $(0, \pm b)$; these points are called the *vertices* of the ellipse. Therefore the interior x-axis has length $2a$ and the interior y-axis has length $2b$ Since $a > b$, the x-axis is called the *major axis* and the y-axis is called the *minor axis*.

Since

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}\leqslant\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant\frac{x^2}{b^2}+\frac{y^2}{b^2},$$

every point on the ellipse, except the vertices $(\pm a, 0)$, is interior to the circle $x^2 + y^2 = a^2$. This circle is called the *major auxiliary* or often just the *auxiliary circle.* And every point on the ellipse, except the vertices $(0, \pm b)$, is exterior to the *minor auxiliary circle* $x^2 + y^2 = b^2$.

Ex. 1. If $2r$ is the length of the interior diameter on the line $y = x \tan \theta$, show that

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}.$$

Show also that, as θ increases from 0 to $\pi/2$, r decreases steadily from a to b.

(iii) Parametric equations.

(*a*) If the point P (x, y) is on the ellipse, then the point Q $(x, ay/b)$ is on the auxiliary circle, P, Q are called *corresponding points* on the two curves (Fig 68). Since every ordinate of the ellipse is b/a times the corresponding ordinate of the auxiliary circle, the ellipse may usefully be regarded as a contracted form of the circle; and all properties of the circle, which are unaffected by such a contraction, hold for the ellipse

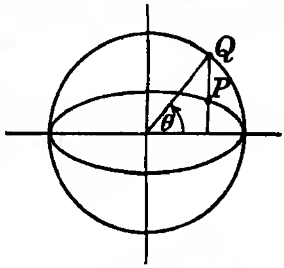


FIG. 68.—ECCENTRIC ANGLE.

Ex. 2. The tangent at P to the ellipse meets the tangent at Q to the auxiliary circle on the major axis

Ex. 3. The area interior to the ellipse is πab.

(*b*) If θ is an angle from the x-axis to the line joining the centre to Q, the co-ordinates of Q are $(a \cos \theta, a \sin \theta)$; and therefore P is $(a \cos \theta, b \sin \theta)$. Thus the ellipse may be represented by the parametric equations

$$x = a \cos \theta, \quad y = b \sin \theta.$$

θ is called an *eccentric angle* of P. The whole real curve is described continuously as θ ranges from 0 to 2π, any two values of θ which differ by an integral multiple of 2π correspond to the same point of the ellipse.

Ex. 4. *If a triangle of maximum area is inscribed in the ellipse, its area is $3\sqrt{3}ab/4$ and its vertices have eccentric angles of the form θ, $\theta + \frac{2}{3}\pi$, $\theta + \frac{4}{3}\pi$. [This may be proved by considering a triangle with vertices at the corresponding points on the auxiliary circle.]*

(*c*) If we put $\tan \theta/2 = t$, the trigonometric parametric equations take the algebraic form

$$x = \frac{a(1 - t^2)}{1 + t^2}, \quad y = \frac{2bt}{1 + t^2}.$$

This parameterisation, in the form of the ratios of quadratic polynomials, is of the type which we saw, in section 20 (*c*), to be characteristic of conics. The equations are equivalent to

$$\left(1 - \frac{x}{a}\right) : \frac{y}{b} : \left(1 + \frac{x}{a}\right) = t^2 : t : 1.$$

Thus the ellipse is the locus of the intersection of the lines

$$\frac{y}{b} = t\left(1 + \frac{x}{a}\right), \quad 1 - \frac{x}{a} = t\frac{y}{b};$$

and these are lines which correspond in a projectivity between the pencils with vertices at the points $A\ (-a, 0)$, $A'\ (a, 0)$.

(iv) **Tangent and polar.**—The following equations come at once from the general theory which precedes this section.

The polar of a point (x_1, y_1), or the tangent at a point (x_1, y_1) on the ellipse, is given by

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

The tangents from an exterior point (x_1, y_1) have the equation

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2.$$

Ex. 5. The equation of the director-circle of the ellipse is

$$x^2 + y^2 = a^2 + b^2.$$

(v) **Conjugate diameters.**—The diameters $y = \lambda x$, $y = \mu x$ are conjugate if and only if

$$\lambda\mu = -b^2/a^2.$$

If P is the point $(a \cos \theta, b \sin \theta)$, the equation of the tangent at P takes the form

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1.$$

The diameter conjugate to OP, O being the centre, is parallel to the tangent at P and therefore has the equation

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 0.$$

It meets the ellipse in points whose eccentric angles are given by the equation in ϕ

$$\cos \phi \cos \theta + \sin \phi \sin \theta = 0,$$

that is by

$$\cos (\phi - \theta) = 0,$$

the roots of which are $\theta + (2k + 1)\pi/2$, where k is any integer. Thus the points with eccentric angles θ and $\theta + \pi/2$ are on conjugate diameters.

Ex. 6. A diameter meets the ellipse at P, P'; and Q is any other point on the ellipse. Prove that PQ, $P'Q$ are parallel to a

pair of conjugate diameters. The interior chords PQ, $P'Q$ are called *supplemental*.

Ex. 7. OP, OQ are conjugate semi-diameters; prove that

$$|OP|^2 + |OQ|^2 = a^2 + b^2.$$

Ex. 8. POP' and QOQ' are two diameters of the ellipse. Prove that the area of the parallelogram, of which these diameters are the diagonals, is greatest when the diameters are conjugate and is then $2ab$. Prove also that the area of the parallelogram formed by the tangents at P, Q, P', Q' is least when the diameters are conjugate and is then $4ab$.

Ex. 9. The equation of the chord joining the points with parameters θ, ϕ is

$$\frac{x}{a}\cos\frac{\theta+\phi}{2} + \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2}.$$

Thus, for a system of parallel chords, $\theta + \phi$ is constant.

(vi) Two ellipses reciprocal with respect to a given ellipse.— The line co-ordinates of the tangent at the point with parameter θ are given by

$$l : m : n = \frac{\cos\theta}{a} : \frac{\sin\theta}{b} : -1.$$

Since $\cos^2\theta + \sin^2\theta = 1$, the equation of the conic-envelope associated with the ellipse is therefore

$$a^2l^2 + b^2m^2 = n^2.$$

The chord joining the points with parameters $\theta + \alpha$, $\theta - \alpha$ has the equation

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos\alpha.$$

This is the equation of the tangent at the point with parameter θ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\alpha.$$

Hence, if θ varies and α remains constant, the chords in question touch a second ellipse.

The pole of the chord, named above, with respect to the original ellipse is the point $(a \sec\alpha \cos\theta,\ b \sec\alpha \sin\theta)$; its locus is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\alpha.$$

Since the points of the third ellipse are the poles for the first ellipse of the tangents of the second ellipse, these ellipses are said to be *reciprocal* with respect to the given ellipse.

Ex. 10. The polar for the first ellipse of the point with parameter θ on the second ellipse is the tangent at the point with parameter θ on the third ellipse.

(vii) Co-normal points.—The equation of the normal at a point (x_1, y_1) on the ellipse is

$$(x - x_1)y_1/b^2 = (y - y_1)x_1/a^2.$$

This line passes through the point (x_0, y_0) if and only if the foot

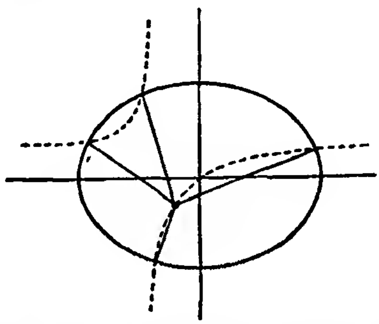


FIG. 69.—THE RECTANGULAR HYPERBOLA OF APOLLONIUS AND THE ASSOCIATED CONCURRENT NORMALS.

of the normal is at an intersection of the ellipse with the rectangular hyperbola (Fig. 69)

$$a^2(x - x_0)y = b^2(y - y_0)x.$$

This rectangular hyperbola, called after *Apollonius*, is a particular case of the rectangular conic described in section 31 (viii).

The t-parameters of the feet of the normals from (x_0, y_0) are therefore given by

$$a^2\left[\frac{a(1-t^2)}{1+t^2} - x_0\right]\left[\frac{2bt}{1+t^2}\right] = b^2\left[\frac{2bt}{1+t^2} - y_0\right]\left[\frac{a(1-t^2)}{1+t^2}\right],$$

which simplifies to

$$t^4 + 2t^3(ax_0 + a^2 - b^2)/(by_0) + 2t(ax_0 - a^2 + b^2)/(by_0) - 1 = 0.$$

This equation, having real coefficients, has 0, 2 or 4 real roots. Actually there are at least two different real roots; for, denoting the real quartic polynomial by $f(t)$, we have $f(0) < 0$, and $f(t) > 0$ for numerically large t; being a continuous function of t, $f(t)$

therefore vanishes for at least one positive and at least one negative value of t Hence, at least two real normals can be drawn from an arbitrary real point. (This result is related to the intuitively obvious fact that the branch of Apollonius' hyperbola which passes through the origin necessarily meets the ellipse in two distinct real points)

Let t_1, t_2, t_3, t_4 be the roots of the quartic when all are real. Then we see that a necessary and sufficient pair of conditions for the normals at the points with these parameters to be concurrent is

$$\Sigma t_1 t_2 = 0, \quad t_1 t_2 t_3 t_4 = -1.$$

If θ_1, θ_2, θ_3, θ_4 are corresponding eccentric angles, we have

$$\tan \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3}{1 - \Sigma t_1 t_2 + t_1 t_2 t_3 t_4};$$

and therefore a necessary (though not sufficient) condition for concurrency of the normals is

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 \equiv \pi \pmod{2\pi}.$$

Ex. 11. Two of the feet of four concurrent normals lie on the line $lx + my + n = 0$. Prove that the other two feet lie on the line

$$\frac{x}{a^2 l} + \frac{y}{b^2 m} - \frac{1}{n} = 0$$

and that the point of concurrence is

$$\left(\frac{(a^2 - b^2)(b^2 m^2 - n^2)l}{(a^2 l^2 + b^2 m^2)n}, \quad \frac{(a^2 - b^2)(n^2 - a^2 l^2)m}{(a^2 l^2 + b^2 m^2)n}\right).$$

[This is most simply done by observing that a conic of the pencil determined by the ellipse and the rectangular hyperbola of Apollonius is the pair of lines in question.]

(viii) Concyclic points. Circle of curvature.

(*a*) The real circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

meets the ellipse in points whose t-parameters are given by

$$(a^2 - 2ga + c)t^4 + 4bft^3 + 2(-a^2 + 2b^2 + c)t^2 + 4bft + (a^2 + 2ga + c) = 0.$$

The coefficients in this quartic being real, the circle has 0, 2 or 4 real, and possibly coincident, intersections with the ellipse. The remarks which follow apply whether the intersections are real or not.

If t_1, t_2, t_3, t_4 are the roots of the quartic, we have $\Sigma t_1 = \Sigma t_1 t_2 t_3$. Conversely, if this relation connects the four parameters, the corresponding points lie on a circle, because then we have effectively just three independent linear equations from which to determine g, f, c.

If the four points on the circle have eccentric angles $\theta_1, \theta_2, \theta_3, \theta_4$, the last relation is equivalent to $\tan \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 0$, or $\theta_1 + \theta_2 + \theta_3 + \theta_4 \equiv 0 \pmod{2\pi}$. In this form it is perhaps simpler to see that, conversely, if $\theta_1 + \theta_2 + \theta_3 + \theta_4 \equiv 0 \pmod{2\pi}$, then the points with these eccentric angles are concyclic. In fact, the circle through the points with angles $\theta_1, \theta_2, \theta_3$ meets the ellipse in a point with angle θ_4', where $\theta_1 + \theta_2 + \theta_3 + \theta_4' \equiv 0 \pmod{2\pi}$. Hence $\theta_4 \equiv \theta_4' \pmod{2\pi}$, so that both θ_4 and θ_4' are eccentric angles of the same point. It is worth noting that this result shows that concyclic points cannot be co-normal, and *vice versa*.

(*b*) Let P, Q, R, S be the four intersections of the circle and ellipse. Any pair of opposite sides of the quadrangle $PQRS$ is given by an equation of the form

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda(x^2 + y^2 + 2gx + 2fy + c) = 0.$$

A pair of parallel lines through the origin is then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \lambda(x^2 + y^2) = 0;$$

the form of this equation shows that the two lines are equally inclined to the axes. This statement applies, with the usual conventions, when two or three of the four points coincide; a fact which is of practical use. Thus, when Q coincides with P, the tangent at P and the line RS are equally inclined to the axes And, when Q, R both coincide with P, the tangent at P and the line PS are equally inclined.

When P, Q, R coincide, the circle, which is uniquely determined, is called the *circle of curvature* at P; and its radius is called the *radius of curvature* of the ellipse at P. The centre of the circle is obviously on the normal at P.

If θ is an eccentric angle of P, the equation of the tangent to the ellipse at P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

Therefore the equation of PS is

$$\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = \cos^2\theta - \sin^2\theta.$$

The circle of curvature belongs to the pencil determined by the ellipse and this pair of lines; therefore its equation is

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$
$$+\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1\right)\left(\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta-\cos^2\theta+\sin^2\theta\right)=0,$$

where λ is determined by

$$\frac{\lambda}{a^2}+\frac{\cos^2\theta}{a^2}=\frac{\lambda}{b^2}-\frac{\sin^2\theta}{b^2}.$$

The equation of the circle is therefore

$$\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$
$$=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1\right)\left(\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta-\cos 2\theta\right).$$

This equation may be rewritten in the form

$$x^2+y^2-2(a^2-b^2)(x/a)\cos^3\theta+2(a^2-b^2)(y/b)\sin^3\theta$$
$$+(a^2-2b^2)\cos^2\theta-(2a^2-b^2)\sin^2\theta=0.$$

The centre of the circle, called the *centre of curvature*, is the point

$$\left(\frac{a^2-b^2}{a}\cos^3\theta,\quad \frac{b^2-a^2}{b}\sin^3\theta\right),$$

and the radius of curvature is $(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}/(ab)$.

Ex. 12. The normal at the point P', with parameter θ', meets the normal at P in the point C. Prove that the limiting position of C as $\theta'\longrightarrow\theta$ is the centre of curvature at P.

[This provides the most rapid method for obtaining the co-ordinates of the centre of curvature. From this point of view, the centre of curvature is the intersection of the normal at P with the line whose equation is obtained by differentiating with regard to θ the equation of the normal. It is the contact-point of the normal in regard to the envelope consisting of all the normals of the ellipse.]

Ex. 13. The locus of the real centres of curvature is the real curve given by

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}.$$

[This may also be described as the locus of points from which two normals to the ellipse coincide. It is known as the *evolute* of the ellipse.]

(ix) Foci, eccentricity.—From the point of view of real geometry, the foci may be introduced by the characteristic property, that every pair of perpendicular lines through a focus is a pair of conjugate lines.

If, then, (x_1, y_1) is a focus, the equations of a pair of perpendicular lines through the point are

$$y - y_1 = m(x - x_1), \quad y - y_1 = -(x - x_1)/m.$$

These are conjugate if and only if

$$a^2 - b^2 = x_1^2 - y_1^2 + (m - 1/m)x_1y_1.$$

This being the case for all values of m, the foci arise as the intersections of the axes with the rectangular hyperbola

$$x^2 - y^2 = a^2 - b^2.$$

Two foci are therefore real, namely the points $F(-(a^2 - b^2)^{1/2}, 0)$, $F'(+(a^2 - b^2)^{1/2}, 0)$. The other two are unreal.

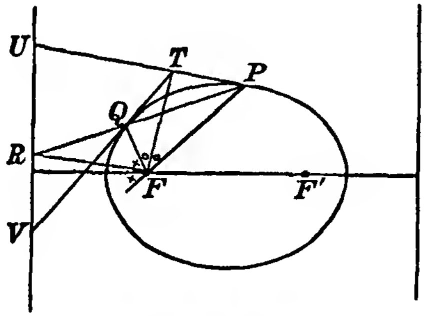


FIG 70.—FOCUS-DIRECTRIX PROPERTIES OF AN ELLIPSE.

It is usual to define a number e (> 0), called the *eccentricity* of the ellipse, by means of the equation

$$a^2e^2 = a^2 - b^2.$$

Evidently $e < 1$. The co-ordinates of F, F' may then be expressed as $(-ae, 0)$, $(ae, 0)$ respectively.

The corresponding directrices are respectively

$$x = -a/e, \quad x = a/e.$$

(x) A geometrical property of the ellipse.—Let T be an exterior point, and let its polar line meet the ellipse at P, Q, and the directrix corresponding to F at R (Fig. 70). Let the tangents TP, TQ meet this directrix in U, V respectively. We prove that FT, FR bisect the angles between FP, FQ and between FU, FV. A similar statement applies in regard to the focus F'.

First observe that the polar of R is FT. Therefore FR, FT are conjugate, and, since F is a focus, perpendicular. Moreover, FR, FT *harm* FP, FQ; therefore FR, FT bisect the angles between FP, FQ.

The second part involves the complex part of the plane, if we are to use a projective argument Let the directrix meet the ellipse at the unreal points L, M. Then the pencil of conics touching TP, TQ at P, Q respectively determines an involution on the directrix in which U, V and L, M are pairs of mates, and R is a double point. The other double point is the intersection of the directrix with the polar of R, namely FT. Therefore FR, FT *harm* FU, FV and therefore bisect the angles between these lines.

An algebraic argument, which proves the statement as well for the parabola and the hyperbola, will be found in the section on polar co-ordinates.

Ex 14. The tangent at P meets the directrix corresponding to F at U. Prove that FP is perpendicular to FU.

(xi) Distance properties.—Let the line through P $(a \cos\theta, b \sin\theta)$, parallel to the major axis, meet the directrix corresponding to F at M and the other at M'. Then

$$\begin{aligned} |FP|^2 &= (a\cos\theta + ae)^2 + b^2\sin^2\theta \\ &= (a + ae\cos\theta)^2, \end{aligned}$$

and therefore, with unsensed lengths,

$$\begin{array}{ll} & |FP| = a + ae\cos\theta. \\ \text{also} & |PM| = a\cos\theta + a/e. \\ \text{Therefore} & |FP| = e|PM|. \\ \text{Similarly} & |F'P| = e|PM'|. \end{array}$$

It follows that

$$|FP| + |F'P| = e(|PM| + |PM'|) = e(2a/e) = 2a.$$

The property $|FP| = e|PM|$, with $e < 1$, is a common starting point for a metrical theory of the ellipse. Some indication of the development of this theory, using polar co-ordinates, will be found in section 36.

The property $|FP| + |F'P| = 2a$ provides a method for drawing the ellipse mechanically. If a loop of string, of total length $2a + 2ae$, be kept taut in the shape of a triangle, with two vertices at F, F' and the pencil at P, the pencil will trace out the ellipse.

(xii) Further properties.—The tangent at P meets the major axis at the point T $(a\sec\theta, 0)$, exterior to the segment FF' (Fig. 71).

Since

$$\frac{|FT|}{|F'T|} = \frac{a \sec \theta + ae}{a \sec \theta - ae} = \frac{a + ae \cos \theta}{a - ae \cos \theta} = \frac{|FP|}{|F'P|},$$

PT bisects the angle from $F'P$ to PF. Therefore the normal at P bisects the angle from FP to PF'.

On FP take H, with P between F and H, so that $|PH| = |PF'|$ Then let $F'H$ meet the tangent at P in Y'. The triangles PHY', $PF'Y'$ are congruent. Therefore FY' is perpendicular to the tangent and Y' is the mid point of F, H.

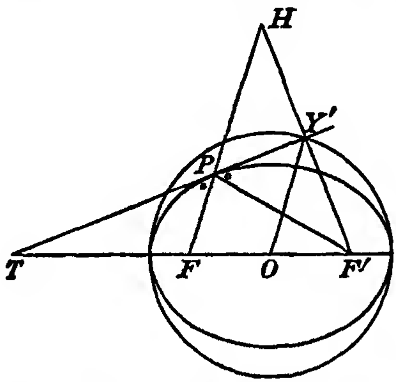


FIG 71 —FURTHER FOCAL PROPERTIES OF AN ELLIPSE

Moreover $|FO| = |OF'|$, therefore OY' is parallel to FH, and $|OY'| = \frac{1}{2}|FH| = a$. Thus the foot of the perpendicular from F' to the tangent at P is on the auxiliary circle; and, similarly, the same is true in regard to the other focus.

Ex. 15. The normal at P meets the major axis at G and the minor axis at K. Prove that $|PG|/|PK| = b^2/a^2$, and that $|PG|^2/(|FP| \, . \, |F'P|) = b^2/a^2$.

34. The hyperbola.

(i) Standard form of the equation.—The standard form for the equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with $a > 0$, $b > 0$.

Many properties of the hyperbola may be immediately inferred from those of the ellipse simply by replacing b^2 by $-b^2$, or b by ib. And indeed there is no distinction at all in regard to projective properties in relation to the containing modified complex plane.

Ex. 1. The equation of the conic-envelope associated with the hyperbola is $a^2l^2 - b^2m^2 = n^2$.

(ii) **Elementary properties.**—We have remarked that the centre is an exterior point. The set of exterior points therefore consists of those for which $x^2/a^2 - y^2/b^2 < 1$; and the set of interior points consists of those for which $x^2/a^2 - y^2/b^2 > 1$. The curve is symmetrical about the centre and each axis (Fig. 72).

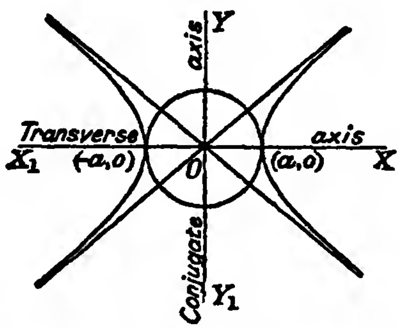


FIG 72.—THE HYPERBOLA WITH ITS ASYMPTOTES AND AUXILIARY CIRCLE.

The hyperbola meets the x-axis in the vertices $A\ (-a, 0)$, $A'\ (a, 0)$, but has no real intersections with the y-axis. The x-axis is called the *transverse* axis and the other the *conjugate* axis.

Since

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} \leqslant \frac{x^2}{a^2} + \frac{y^2}{a^2},$$

every point of the hyperbola, except A, A', is exterior to the *auxiliary circle* $x^2 + y^2 = a^2$. The curve is in fact unbounded; for to every value of $x^2 > a^2$, there correspond two real values of y. The curve has no real point for which $x^2 < a^2$.

The asymptotes are real; their equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

or, separately,

$$\frac{x}{a} + \frac{y}{b} = 0, \quad \frac{x}{a} - \frac{y}{b} = 0.$$

The asymptotes are perpendicular if and only if $a = b$; the conic is then a rectangular hyperbola, and its equation takes the form

$$x^2 - y^2 = a^2.$$

Since, for any point on the hyperbola,

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 < \frac{x^2}{a^2},$$

the whole curve lies within that angle between the asymptotes which contains the x-axis.

(iii) Parametric equations.

(a) The parameterisation of the hyperbola by means of real singly-periodic (trigonometric) functions of a real eccentric angle is not possible in the same way as for the ellipse. However, a correspondence between the hyperbola and the auxiliary circle may be set up as follows.

Let Q be a point on the auxiliary circle such that θ is an angle from the x-axis to OQ. Let the tangent at Q meet the x-axis at N. The ordinate at N meets the hyperbola in two real points; we choose that point P which is on the same side of the x-axis as Q. Then the co-ordinates of P are given by

$$x = a \sec \theta, \; y = b \tan \theta.$$

As θ varies from $-\pi/2$ to $+\pi/2$, P describes continuously the whole part of the hyperbola on the positive side of the y-axis, and as θ varies from $\pi/2$ to $3\pi/2$, P describes continuously the whole part of the hyperbola on the negative side of the y-axis These parts are called the *branches* of the curve

Putting $t = \tan \theta/2$, we obtain an algebraic parameterisation given by

$$\frac{x}{a} = \frac{1 + t^2}{1 - t^2}, \quad \frac{y}{b} = \frac{2t}{1 - t^2}.$$

These equations give

$$\frac{x}{a} - 1 : \frac{y}{b} : \frac{x}{a} + 1 = t^2 \cdot t \,.\, 1.$$

The hyperbola is therefore the locus of the intersection of the lines

$$\left(\frac{x}{a} - 1\right) = t\left(\frac{y}{b}\right), \quad \frac{y}{b} = t\left(\frac{x}{a} + 1\right);$$

and these lines correspond in a projectivity between the pencils with vertices at A', A respectively.

Ex. 2. The distances from the asymptotes of the point (x, y) on the hyperbola are p and q; prove that $pq = a^2b^2/(a^2 + b^2)$ and that either p or q tends to zero as $|x|$ or $|y|$ increases without limit.

(b) An alternative parameterisation, based on the identity $\cosh^2\phi - \sinh^2\phi = 1$, is

$$x = a\cosh\phi, \quad y = b\sinh\phi,$$

for the branch on the positive side of the y-axis, and

$$x = -a\cosh\phi, \quad y = -b\sinh\phi,$$

for the other branch. The whole of each branch is described continuously as ϕ varies from $-\infty$ to $+\infty$.

(c) A parameterisation, involving an unreal function of a real angle ψ, which is often favoured because of its analogy with the parameterisation of the ellipse, is

$$x = a\cos\psi, \quad y = ib\sin\psi.$$

There is a (1, 1) correspondence between the point $(a\cos\psi, ib\sin\psi)$ on the hyperbola and the point $(a\cos\psi, b\sin\psi)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$, and so also between the point on the hyperbola and the point $(a\cos\psi, a\sin\psi)$ on the auxiliary circle $x^2 + y^2 = a^2$, common to the hyperbola and the ellipse.

(iv) Statement on particular allied loci.—The following formulae may be inferred at once from the corresponding formulae for an ellipse; they are derived by substituting $-b^2$ for b^2 in the formulae relating to the ellipse.

The equation of the polar of a point (x_1, y_1) or of the tangent at a point (x_1, y_1) on the hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

The equation of the tangents from an exterior point (x_1, y_1) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^{\,2}}{a^2} - \frac{y_1^{\,2}}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2.$$

The equation of the director circle is

$$x^2 + y^2 = a^2 - b^2.$$

This circle is real only if $a > b$. In the case when the hyperbola is rectangular, the director circle is the point-circle at the origin.

The diameters $y = \mu x$, $y = \mu' x$ are conjugate if and only if $\mu\mu' = b^2/a^2$.

Ex. 3. The asymptotes of a rectangular hyperbola bisect the angles between any pair of conjugate diameters.

Ex. 4 P is on one and Q is on the other of the *conjugate* hyperbolas $x^2/a^2 - y^2/b^2 = \pm 1$, and OP, OQ are mates in the common involution of conjugate diameters; prove that the tangents at P and Q to their respective hyperbolas meet on a common asymptote.

Ex. 5. The rectangular hyperbola of Apollonius, which determines the feet of the normals from a point (x_0, y_0), has the equation

$$a^2(x - x_0)y + b^2(y - y_0)x = 0.$$

Prove that there are two or four real normals through every real point, allowing for possible coincidences

Ex. 6. If the line $lx + my + n = 0$ contains two of the feet of four concurrent normals, prove that the other two lie on the line

$$\frac{x}{a^2l} - \frac{y}{b^2m} - \frac{1}{n} = 0;$$

and find the co-ordinates of the point of concurrence.

(v) Concyclic points.—Just as in the case of an ellipse, we can prove that a real circle meets the hyperbola in 0, 2 or 4 real points. In the case of four points, the real lines joining these in pairs are equally inclined to the axes.

Ex. 7. The equation of the circle of curvature at the point $(a \sec \theta, b \tan \theta)$ is

$$\left(\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}\right)\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\bigg/\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$
$$= \left(\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1\right)\left(\frac{x}{a}\sec\theta + \frac{y}{b}\tan\theta - \sec^2\theta - \tan^2\theta\right)$$

or

$$x^2 + y^2 - 2(a^2 + b^2)(x/a)\sec^3\theta + 2(a^2 + b^2)(y/b)\tan^3\theta + (a^2 + 2b^2)\sec^2\theta + (2a^2 + b^2)\tan^2\theta = 0.$$

Find the co-ordinates of the centre of curvature and the radius of curvature

(vi) Foci, eccentricity.—As for the ellipse, the foci of the hyperbola are at the intersections of the axes with the rectangular hyperbola

$$x^2 - y^2 = a^2 + b^2.$$

Two foci are therefore real and two unreal. The real foci are the points $F(-ae, 0)$, $F'(ae, 0)$, where the *eccentricity* e is defined by

$$a^2e^2 = a^2 + b^2.$$

The corresponding directrices are respectively

$$x = -a/e, x = a/e.$$

Since $e > 1$, the directrices are exterior to the hyperbola.

Ex. 8. Consider section 33 (x) in relation to the hyperbola.

Ex. 9. The eccentricity of a rectangular hyperbola is $\sqrt{2}$.

(vii) Distance properties.—Let the line through a point P on the hyperbola parallel to the transverse axis meet the directrix $x = -a/e$ in M and the other directrix in M'. Then we may prove, as for the ellipse, that

$$|FP| = e|PM|, \ |F'P| = e|PM'|,$$

and that, if P is on the positive side of the y-axis.

$$|FP| - |F'P| = 2a,$$

whereas, if P is on the negative side,

$$|F'P| - |FP| = 2a.$$

And we may also show that the tangent at P is the internal bisector of the angle FPF'; and that the feet of the perpendiculars from the foci onto the tangent lie on the auxiliary circle (Fig. 73).

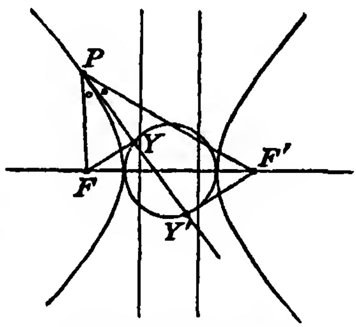


FIG. 73.—FOCAL PROPERTIES OF THE HYPERBOLA.

(viii) Equation of the hyperbola referred to the asymptotes as axes of co-ordinates.—Using oblique axes, rectangular in the case of a rectangular hyperbola, the equation of the hyperbola referred to its asymptotes has the form $xy = k$. If we choose the positive senses on these axes suitably, we may arrange for k to be positive. Putting $k = c^2$, the equation becomes

$$xy = c^2.$$

The curve may then be parameterised by means of the equations

$$x = ct, \ y = c/t.$$

The equation of the tangent at the point with parameter t is

$$x/t + yt = 2c.$$

This meets the asymptotes at the points $(2ct, 0)$ and $(0, 2c/t)$; the point of contact is therefore the mid point of these two. The line co-ordinates of the tangent are given by

$$l : m : n = 1/t : t : -2c.$$

The equation of the conic-envelope associated with the hyperbola therefore becomes

$$n^2 = 4c^2lm.$$

35. The real parabola.

(i) Standard form of the equation.—The positive senses are chosen on the axes of co-ordinates so that the equation has the form

$$y^2 = 4ax$$

with $a > 0$.

It is easily verified that the polar of the point $(-a, 0)$ has two real intersections with the parabola. The exterior of the parabola is therefore the set of points for which $y^2 - 4ax > 0$, and the interior is the set for which $y^2 - 4ax < 0$. The curve is symmetrical about the x-axis (Fig. 74).

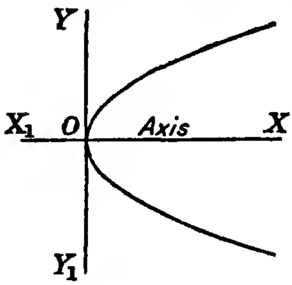


FIG 74.—THE REAL PARABOLA

(ii) Parametric equations.—The curve admits the very simple parametric equations

$$x = at^2,\ y = 2at.$$

It is generated by the point common to the lines

$$2x = ty,\ y = 2at$$

which correspond in a projectivity between the pencil with vertex at the vertex $(0, 0)$ of the parabola and the pencil of parallel diameters.

(iii) Tangent.—The equation of the tangent at the point with parameter t is

$$yt = x + at^2.$$

The line co-ordinates of the tangent are given by

$$l : m : n = 1 : -t : at^2;$$

therefore the equation of the associated conic-envelope is

$$am^2 = nl.$$

Ex. 1. Perpendicular tangents meet on the director-line $x + a = 0$.

Ex. 2. The equation of the chord joining the points with parameters t_1, t_2 is

$$\tfrac{1}{2}y(t_1 + t_2) = x + at_1t_2,$$

and the pole of this line is the point $(at_1t_2,\ a(t_1 + t_2))$.

Ex. 3 The mid points of the interior chords parallel to $y = \mu x$ lie on the diameter $y = 2a/\mu$.

(iv) Normal.—The equation of the normal at the point P with parameter t is

$$xt + y = at^3 + 2at.$$

This line meets the x-axis at the point $N(at^2 + 2a,\ 0)$. The difference between the abscissae of N and P is therefore $2a$, which is constant; this distance is called the *subnormal*.

The parameters of the feet of the normals which are concurrent at the point (x_0, y_0) are the roots t_1, t_2, t_3 of the cubic

$$at^3 + (2a - x_0)t - y_0 = 0.$$

There is always at least one real root. Evidently

$$t_1 + t_2 + t_3 = 0.$$

Ex. 4. A curve whose subnormal is constant, for all points on the curve, is a parabola.

Ex. 5. If $t_1 + t_2 + t_3 = 0$, the normals at the points with parameters t_1, t_2, t_3 are concurrent. Find the co-ordinates of the point of concurrence

Ex. 6. The feet of the normals from (x_0, y_0) are at the accessible intersections of the parabola with the rectangular hyperbola $xy + (2a - x_0)y - 2ay_0 = 0$.

Ex 7 The locus of the intersection of the normals at the extremities of a variable chord of given gradient is a normal.

(v) Concyclic points.—The real circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

meets the parabola in the points whose parameters are given by

$$a^2t^4 + 2a(2a + g)t^2 + 4fat + c = 0.$$

This equation has 0, 2 or 4 real roots, possibly with coincidences If t_1, t_2, t_3, t_4 are the roots, we have

$$t_1 + t_2 + t_3 + t_4 = 0.$$

Conversely, there is a circle through four points with parameters t_1, t_2, t_3, t_4 such that $t_1 + t_2 + t_3 + t_4 = 0$. For the circle through the first three of these points meets the parabola again in the point whose parameter is $-(t_1 + t_2 + t_3)$ and this is the same as the point with parameter t_4.

Ex. 8. A circle through any three co-normal points passes through the vertex of the parabola.

Ex. 9. The equation of the circle of curvature at the point with parameter t is

$$(1 + t^2)(y^2 - 4ax) + (x - yt + at^2)(x + yt - 3at^3) = 0.$$

The centre of curvature is the point $(a(2 + 3t^2), -2at^3)$ and the radius of curvature is $2a(1 + t^2)^{3/2}$.

Ex. 10. The centre of curvature at the point with parameter t is on the normal with parameter $-2t$.

(vi) Focus.—We have seen that the parabola has just one focus. Let this be (x_1, y_1). The equations of two perpendicular lines through this are

$$y - y_1 = m(x - x_1), \quad y - y_1 = -(x - x_1)/m.$$

These lines are conjugate if and only if

$$2a = (m - 1/m)y_1 + 2x_1.$$

This being the case for all values of m, we have $x_1 = a$, $y_1 = 0$. The focus is therefore the point $F(a, 0)$.

The directrix is consequently the line $x + a = 0$, which has already arisen as the director-line.

Let the diameter through any point P (x, y) of the parabola meet the directrix at M. Then $|FP| = |PM| = x + a$. The property $|FP|/|PM| =$ constant is analogous to properties of the ellipse and hyperbola. Since the constant is 1 in the present case, we say that the *eccentricity* of the parabola is 1.

Ex. 11. The polar of a point R on the directrix is the line through F perpendicular to RF. If it meets the parabola at P, Q and the directrix at G, prove that PR, QR are the bisectors of the angles between FR, RG.

Ex. 12. The foot of the perpendicular from the focus to any tangent lies on the tangent at the vertex of the parabola.

(vii) The orthocentre of a triangle of tangents is on the directrix (Fig. 75).—The algebraic proof is straightforward. It is interesting also to deduce the theorem from that of Brianchon. Let A, B, C be the tangents, D the directrix, I the inaccessible line. From the exterior point B . D a second tangent B′ is drawn, and from C . D a second tangent C′; then B is perpendicular to B′ and C to C′. Applying Brianchon's theorem to the hexalateral (AB′C′, ICB), we find that the lines joining A . B to I . C′, A . C to I . B′, B . B′ to C . C′ are concurrent. The first two lines are altitudes of the triangle of tangents and the third is the directrix D. The theorem is Steiner's.

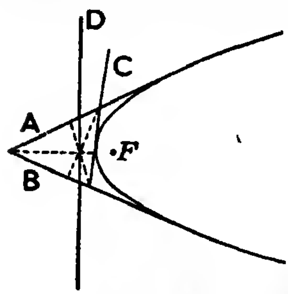


FIG. 75.—THE ORTHOCENTRE OF A TRIANGLE OF TANGENTS IS ON THE DIRECTRIX.

(viii) The circumcircle of a triangle of tangents passes through the focus. (Fig. 76).

(*a*) Let the points of contact of the tangents have parameters t_1, t_2, t_3. Then the equation of the circumcircle is

$$\frac{\lambda}{x - yt_1 + at_1^2} + \frac{\mu}{x - yt_2 + at_2^2} + \frac{\nu}{x - yt_3 + at_3^2} = 0$$

with
$$\lambda + \mu + \nu = \lambda t_2 t_3 + \mu t_3 t_1 + \nu t_1 t_2,$$
$$\lambda(t_2 + t_3) + \mu(t_3 + t_1) + \nu(t_1 + t_2) = 0.$$

Therefore

$$\lambda : \mu : \nu = (1 + t_1^2)(t_2 - t_3) : (1 + t_2^2)(t_3 - t_1) : (1 + t_3^2)(t_1 - t_2).$$

The statement is now obvious.

(*b*) The theorem may also be inferred from the fact that, since the triangle FIJ is also circumscribed about the parabola, there is a conic through the vertices of the two triangles. As it passes through the circular points I, J, this conic is the circumcircle in question.

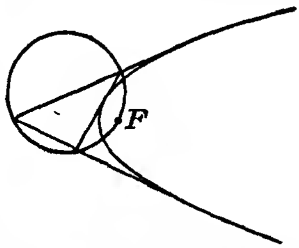


FIG. 76.—THE CIRCUMCIRCLE OF A TRIANGLE OF TANGENTS PASSES THROUGH THE FOCUS

(*c*) Five lines, of which no three are concurrent, determine a unique irreducible conic-envelope. Therefore one parabola touches the sides of a given quadrilateral of accessible lines. Taking the sides in threes, it follows that the four circumcircles of the triangles so formed have a point, called the *Miquel point* of the quadrilateral, in common, namely the focus of the parabola.

Ex 13. The orthocentres of the four triangles formed by the sides of a quadrilateral, taken in threes, lie on a line, and the feet of the perpendiculars from the Miquel point to the sides lie on the line midway between the first line and the Miquel point.

36. Focal properties of a real conic : polar co-ordinates.

(i) Polar co-ordinates.—A number of interesting properties of a real conic may be derived from the property

$$|FP|/|PM| = e,$$

already noted. Polar co-ordinates are the most suitable.

Some preliminary remarks are necessary. If we take the pole and initial line of the system of polar co-ordinates r, θ to be the

origin and x-axis respectively (Fig. 77), the distance co-ordinates x, y are related to r, θ by the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

While the distance co-ordinates of P are unique (for a given scale of measurement) the polar co-ordinates of P are not unique In fact, if (r, θ) are polar co-ordinates of P, so also are $(r, \theta + 2k\pi)$ and $(-r, \theta + (2k+1)\pi)$, where k is any positive or negative integer or zero.

Every accessible line has an equation of the form

$$lx + my + n = 0,$$

with l, m not both zero. In polar co-ordinates this becomes

$$r(l \cos \theta + m \sin \theta) + n = 0.$$

Thus every accessible line has an equation of the form

$$u \cos \theta + v \sin \theta = w/r,$$

with u, v not both zero. The converse is obviously true.

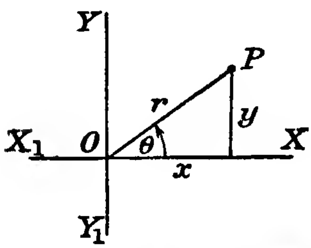


FIG 77 —POLAR CO-ORDINATES.

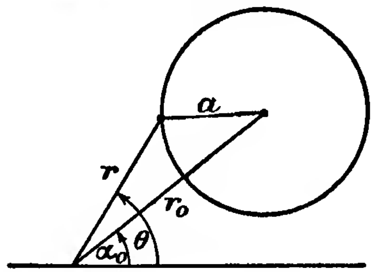


FIG. 78 —FIGURE RELATING TO THE POLAR EQUATION OF A CIRCLE.

Equivalent forms for the equation of a line in polar co-ordinates are easily seen to be

$$A \cos (\theta + \alpha) = w/r, \quad B \sin (\theta + \beta) = w/r,$$

and, with a redundant parameter,

$$C \cos \theta + D \cos (\theta - \alpha) = w/r.$$

The equation of a circle of radius a with centre at the origin is obviously $r = a$ If, however, the centre is at the point (r_0, α_0) (Fig. 78), the equation, derived from the extended form of Pythagoras' theorem, is

$$r^2 - 2rr_0 \cos (\theta - \alpha_0) + r_0{}^2 = a^2,$$

reducing to the simple form

$$r = 2a \cos (\theta - \alpha_0)$$

when the pole is on the circumference $(r_0 = a)$.

Ex. 1. The tangent at the point (r_1, θ_1) on the circle

$$r^2 - 2rr_0 \cos(\theta - \alpha_0) + r_0^2 = a^2$$

has the equation

$$r_1 \cos(\theta - \theta_1) - r_0 \cos(\theta - \alpha_0) = (a^2 + r_0 r_1 \cos(\theta_1 - \alpha_0) - r_0^2)/r.$$

(ii) Polar equation of a real conic.—Consider now a conic, with focus F and associated directrix D (Fig. 79). We take F as pole and the focal axis as initial line, measuring θ so that, if X is the foot of the perpendicular from F to D, the angular co-ordinate of X is π. Let the ordinate through F meet the conic at L, L'; LL' is called the *latus rectum* of the conic and we denote $|LL'|$ by 2λ. Let P be a general point on the conic, with co-ordinates (r, θ), and let PM, LN be the perpendiculars from P, L to D For an ellipse or parabola, we restrict r to be positive; then

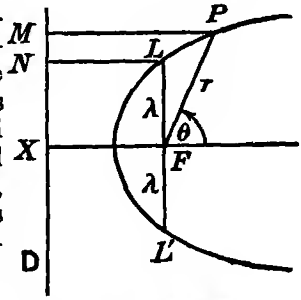


FIG. 79.—FIGURE RELATING TO THE POLAR EQUATION OF A CONIC.

$$|FP|/|PM| = |FL|/|LN| = e,$$

and so, first,

$$|FX| = |LN| = \lambda/e$$

Therefore

$$|PM| = |FX| + r\cos\theta = \lambda/e + r\cos\theta,$$

and

$$r = e(\lambda/e + r\cos\theta).$$

The equation of the conic is therefore

$$\lambda/r = 1 - e\cos\theta.$$

In the case of a hyperbola, let $2\theta_0$ be the magnitude of that angle between the asymptotes which contains the transverse axis, and let k be any positive or negative integer or zero. If P is on the branch of the hyperbola which lies between D and F, we assign to P polar co-ordinates (r, θ) such that $r > 0$ and $2k\pi + \theta_0 < \theta < 2k\pi + 2\pi - \theta_0$ And, if P is on the other branch, we assign to P polar co-ordinates (r, θ) such that $r < 0$ and $2k\pi_0 - \theta < \theta < 2k\pi + \theta_0$.

Then, as above, and with the same definition of latus rectum, we find that these co-ordinates of P satisfy the equation already found, namely $\lambda/r = 1 - e\cos\theta$.

Ex. 2. If the focus is the pole and the focal axis is the line $\theta = \omega$, prove that the equation of the conic is

$$\lambda/r = 1 - e\cos(\theta - \omega).$$

(iii) Equations of chord and tangent.—The equation of a chord

is most conveniently obtained when its intersections with the conic are given with angular co-ordinates in the form $\alpha \pm \beta$.

By part (i), the equation of the chord may be put in the form

$$\lambda/r = C\cos\theta + D\cos(\theta - \alpha).$$

C, D are determined by the equations

$$1 - e\cos(\alpha + \beta) = C\cos(\alpha + \beta) + D\cos\beta,$$
$$1 - e\cos(\alpha - \beta) = C\cos(\alpha - \beta) + D\cos\beta.$$

Therefore $$C = -e, D = \sec\beta.$$

Thus the equation of the chord is

$$\lambda/r = \sec\beta\cos(\theta - \alpha) - e\cos\theta.$$

Now let $\beta \longrightarrow 0$. Then the limit of the chord is the tangent at the point with angular co-ordinate α, and its equation is

$$\lambda/r = \cos(\theta - \alpha) - e\cos\theta.$$

The tangents at the points with angular co-ordinates α_1, α_2 meet on the line given by

$$\cos(\theta - \alpha_1) = \cos(\theta - \alpha_2)$$

Thus, the tangents meet on one of the bisectors of the angles between the lines joining the focus to the points of contact.

Ex. 3. The tangent at the point with angular co-ordinate α meets the directrix in the point for which $\theta = \alpha + \pi/2$. Thus the polar of a point on the directrix is the perpendicular through F to the line joining the point to F.

(iv) Equation of the polar of a point.—The points of contact of the tangents drawn from an exterior point (r_1, θ_1) have angular co-ordinates of the form $\theta_1 \pm \beta$, as we have just seen. Since the tangent at the point with angular co-ordinate $\theta_1 + \beta$ contains (r_1, θ_1), β is determined by the equation

$$\lambda/r_1 = \cos\beta - e\cos\theta_1.$$

Therefore, by part (iii), the equation of the polar of (r_1, θ_1) is

$$(e\cos\theta + \lambda/r)(e\cos\theta_1 + \lambda/r_1) = \cos(\theta - \theta_1).$$

Ex. 4. A variable chord, joining the points with angular co-ordinates $\alpha \pm \beta$, subtends a fixed angle 2β at the focus. Prove that the pole of the chord lies on the fixed conic

$$(\lambda\sec\beta)/r = 1 - e\sec\beta\cos\theta,$$

and that the chord touches the fixed conic

$$(\lambda\cos\beta)/r = 1 - e\cos\beta\cos\theta$$

at the point with angular co-ordinate α.

Ex. 5. If two tangents to a parabola meet at a constant angle, their point of intersection is, in general, on a hyperbola with the same focus and directrix as the parabola. Discuss the case where the angle is a right angle.

Ex. 6. PFP', QFQ' are two perpendicular focal chords of a conic. Prove that

$$\frac{1}{|PF|\,.\,|FP'|}+\frac{1}{|QF|\,.\,|FQ'|}$$

is constant.

Ex. 7. A circle of constant radius passes through the focus of a conic and meets the conic at the points P, Q, R, S. Prove that

$$|FP|\,.\,|FQ|\,.\,|FR|\,.\,|FS|$$

is constant.

37. Conics in relation to a triangle: trilinear co-ordinates.

Trilinear co-ordinates were defined in section 11 (iv) for a modified real euclidean plane. The restriction to a real plane is, however, unnecessary; the essential thing in the definition is that the unit point is to be so chosen that, in the case of real accessible points, the homogeneous co-ordinates are proportional to the distances of the points from the sides of a real triangle of reference. In this section, therefore, we suppose that the plane is a modified complex euclidean plane: properties of the real plane may be inferred as particularities.

(i) Equation of a circle.—The equation of the circumcircle of a real triangle of reference ABC necessarily has the form

$$f\beta\gamma+g\gamma\alpha+h\alpha\beta=0.$$

The equations of the tangents at A, B, C are respectively

$$\frac{\beta}{g}+\frac{\gamma}{h}=0,\quad \frac{\gamma}{h}+\frac{\alpha}{f}=0,\quad \frac{\alpha}{f}+\frac{\beta}{g}=0.$$

From the geometry of the real part of the circle, these equations are seen independently to be respectively the same as

$$\frac{\beta}{b}+\frac{\gamma}{c}=0,\quad \frac{\gamma}{c}+\frac{\alpha}{a}=0,\quad \frac{\alpha}{a}+\frac{\beta}{b}=0,$$

where a, b, c denote the real distances $|BC|$, $|CA|$, $|AB|$. Therefore

$$f:g:h=a:b:c,$$

and the equation of the circumcircle is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0$$

or

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=0.$$

The equation of any circle is of the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma) = 0.$$

This is because the circle belongs to the pencil determined by the circumcircle of ABC and the pair of lines consisting of the accessible line together with that line which joins the accessible common points of the two circles, if there are accessible common points: If there are no accessible common points, the pair of lines is to be taken to be the inaccessible line counted twice, in which case $l : m : n = a : b : c$, and the circle is concentric with the circumcircle.

(ii) The conic-envelope (I, J).—The circular points are those given by the equations

$$\begin{aligned} a\beta\gamma + b\gamma\alpha + c\alpha\beta &= 0, \\ a\alpha + b\beta + c\gamma &= 0. \end{aligned}$$

If l, m, n are co-ordinates of a line through I or J, then these two equations hold simultaneously with

$$l\alpha + m\beta + n\gamma = 0.$$

From the last two, we have

$$\alpha : \beta : \gamma = bn - cm \,.\, cl - an \,.\, am - bl,$$

and therefore the equation of the conic-envelope (I, J) is

$$\frac{a}{bn - cm} + \frac{b}{cl - an} + \frac{c}{am - bl} = 0.$$

This equation may be rearranged and expressed in the form

$$l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C = 0,$$

the cosines referring to the angles A, B, C of the triangle of reference. (Note that $a^2 = b^2 + c^2 - 2bc \cos A$, *etc*.)

(iii) Circle with given centre and radius.—We find the equation of the conic-envelope associated with a circle of given real centre and real radius.

Let the distances of the centre from BC, CA, AB be respectively α_0, β_0, γ_0, taking account of sign; and let the radius be r.

The conic-envelope belongs to the range determined by the pencil with vertex at $(\alpha_0, \beta_0, \gamma_0)$ counted twice and by the conic-envelope (I, J). Therefore its equation has the form

$$\begin{aligned} &(l\alpha_0 + m\beta_0 + n\gamma_0)^2 \\ &\qquad = k(l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C). \end{aligned}$$

To find k, we observe that a particular tangent of the circle, parallel to BC, is (*cf*. *Ex*. 1 of section 11 (iv))

$$2\Delta\alpha = (\alpha_0 + r)(a\alpha + b\beta + c\gamma),$$

where Δ denotes the area of the real triangle ABC. Expressing the fact that this line belongs to the conic-envelope, we find without difficulty that $k = r^2$. Therefore the equation sought is

$$(l\alpha_0 + m\beta_0 + n\gamma_0)^2 = r^2(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C).$$

(iv) Rectangular conics.—We prove next that a necessary and sufficient condition for the conic

$$p\alpha^2 + q\beta^2 + r\gamma^2 + 2u\beta\gamma + 2v\gamma\alpha + 2w\alpha\beta = 0$$

to be rectangular is

$$p + q + r - 2u\cos A - 2v\cos B - 2w\cos C = 0.$$

Let I, J have co-ordinates (necessarily not distances) $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$. Then there is a number $k \neq 0$ such that, for all l, m, n,

$$(l\alpha_1 + m\beta_1 + n\gamma_1)(l\alpha_2 + m\beta_2 + n\gamma_2) = k(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C).$$

Therefore $$\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2 = k,$$

$$\beta_1\gamma_2 + \beta_2\gamma_1 = -2k\cos A, \quad \gamma_1\alpha_2 + \gamma_2\alpha_1 = -2k\cos B, \quad \alpha_1\beta_2 + \alpha_2\beta_1 = -2k\cos C.$$

The given conic is rectangular if and only if I, J are conjugate points; that is, if and only if

$$p\alpha_1\alpha_2 + q\beta_1\beta_2 + r\gamma_1\gamma_2 + u(\beta_1\gamma_2 + \beta_2\gamma_1) + v(\gamma_1\alpha_2 + \gamma_2\alpha_1) + w(\alpha_1\beta_2 + \alpha_2\beta_1) = 0.$$

Substituting the expressions just found, and dividing by k, this condition becomes the one stated above.

The condition is important on account of its linearity. From this it follows at once that if two conics are rectangular, so is every conic of the pencil which they determine. (*Cf* section 31 (i) *Ex.* 3.)

Ex. 1. The lines $l_1\alpha + m_1\beta + n_1\gamma = 0$, $l_2\alpha + m_2\beta + n_2\gamma = 0$ are perpendicular if and only if

$$l_1l_2 + m_1m_2 + n_1n_2 - (m_1n_2 + m_2n_1)\cos A - (n_1l_2 + n_2l_1)\cos B - (l_1m_2 + l_2m_1)\cos C = 0.$$

Ex. 2. Every conic through the vertices and orthocentre of a triangle is rectangular. So is every conic through the centres of the inscribed and escribed circles.

Ex. 3. The equation of the nine-point circle of the triangle of reference is

$$(-a\alpha + b\beta + c\gamma)\alpha\cos A + (a\alpha - b\beta + c\gamma)\beta\cos B + (a\alpha + b\beta - c\gamma)\gamma\cos C = 0.$$

(v) Conic with given opposite foci.

(*a*) Let us suppose that two opposite foci F $(\alpha_1, \beta_1, \gamma_1)$, F' $(\alpha_2, \beta_2, \gamma_2)$ of a central conic are given. This is equivalent to being given four tangents to the conic, namely FI, FJ, $F'I$, $F'J$. The associated conic-envelope therefore belongs to the range determined by these four lines. Two members of the range are the conic-envelope (F, F'), whose equation is

$$(l\alpha_1 + m\beta_1 + n\gamma_1)(l\alpha_2 + m\beta_2 + n\gamma_2) = 0,$$

and the conic-envelope (I, J), whose equation is

$$l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C = 0.$$

The envelope associated with the central conic has therefore an equation of the form

$$(l\alpha_1 + m\beta_1 + n\gamma_1)(l\alpha_2 + m\beta_2 + n\gamma_2) + k(l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C) = 0.$$

(*b*) If a parabola has given focus $(\alpha_1, \beta_1, \gamma_1)$ and touches the inaccessible line at $(\alpha_2, \beta_2, \gamma_2)$, so that $a\alpha_2 + b\beta_2 + c\gamma_2 = 0$, the last equation in (*a*) may similarly be proved to represent the conic-envelope associated with the parabola.

Further, if the parabola is inscribed in the triangle of reference, we have

$$\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2 = -k,$$

and therefore

$$\frac{a}{\alpha_1} + \frac{b}{\beta_1} + \frac{c}{\gamma_1} = 0.$$

Thus we have proved again that the circumcircle of a triangle formed by three tangents of a parabola contains the focus of the parabola.

Ex. 4. The circumcircle of a triangle self-polar with respect to a rectangular conic contains the centre of the conic.

38. The projective aspect of metrical theorems: Significance of diagrams.

In euclidean geometry, the real euclidean plane E_R is explored by metrical methods, that is by considerations of distance and angle in which the ideas of congruence of lengths and congruence of angles are fundamental. In consequence, the statements of the theorems in euclidean geometry are themselves largely metrical (as in the theorem that the angles opposite to the equal sides of an isosceles triangle are equal) and their proofs may also

be metrical, even when the facts asserted are purely properties of incidence (as in Desargues' theorem).

It is remarkable that metrical notions, of congruence in particular, in E_R can be replaced by equivalent projective relations concerning the circular points, here I', J', and their join l′ in the covering modified complex plane E_M. Thus, let $A, \ldots$ and A, . . . denote points and lines in E_R and let the lines in E_M which cover $AB, \ldots$ and A, . . . be denoted by $(AB)', \ldots$ and A′, ... Then, the statements for E_R:

(i) A is parallel to B;
(ii) A is perpendicular to B;
(iii) angle {A, B} = angle {C, D} = − angle {E, F};
(iv) $|AB| = |CD|$

are equivalent respectively to the statements for E_M (where the point A′ . l′ is denoted by A', and so on):

(i) A′ . B′ is on l′;
(ii) A', B' *harm* I', J';
(iii) $\{A', B';\ I'\ J'\} = \{C', D';\ I', J'\} = \{F', E';\ I', J'\}$;
(iv) there is a point F in E_R such that $(AB)'$ meets $(CF)'$ on l′, $(AF)'$ meets $(CB)'$ on l′ and such that D, F lie on a conic through I', J' for which C is the pole of l′ (*i e.* $ABCF$ is a parallelogram and F, D are on a circle with centre at C).

If now K is any figure of points, lines, conics, conic-envelopes, . . . in E_R and if K′ is the covering figure in E_M, any metrical theorem T about K is equivalent to a theorem T′ about K′, I', J', l′ in which the relations between the elements of K′ and I', J', l′ are projective. We may call T′ the *projective aspect* of T

Theorem T′ may be important for several reasons. It may give the reader a new understanding of the content of theorem T. It may have intrinsic interest, and then one would naturally seek a proof by projective methods. And a certain coherence is given to the body of metrical theorems when it is realised that the projective aspects of several different metrical theorems may be projectively equivalent (a relationship explained in Chapter VI); these several theorems can then be settled by a single proof (*cf.* section 30, *Exs.* 4–7). But it should be realised that what may be expressible simply in metrical terms need not be simply expressible in projective terms, and *vice versa*, since, for example, congruence of distances is not a simple projective idea (nor is it projectively invariant).

In illustration of what has been said, let T be the theorem that if ABC is a triangle in E_R with $|AB| = |AC|$, then angle $\{AB, BC\}$ = angle $\{CB, CA\}$.

In E_M, let $(BC)'$, $(CA)'$, $(AB)'$ meet l' in D', E', F' respectively (Fig. 80); then T$'$ is the theorem that if ABC is a real triangle such that B, C both lie on a conic through I', J' and having A as the pole of l', then $\{D', F', I', J'\} = \{E', D'; I', J'\}$. The reader should have no difficulty in finding a projective proof of this theorem with the help of the remark that D' is a double point of the involution on l' in which I', J' and E', F' are pairs of mates.

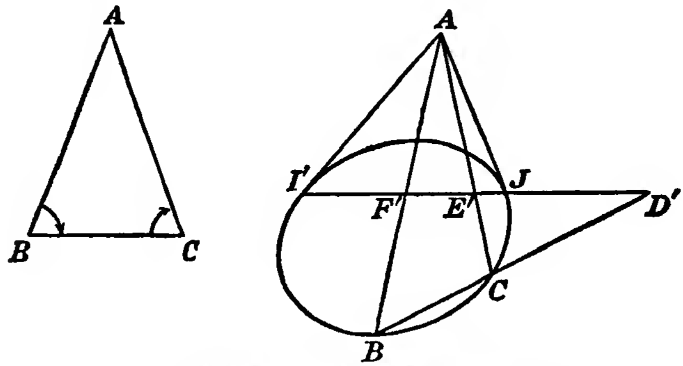


FIG 80—FIGURE FOR THE THEOREM OF SECTION 38 (a).

More generally, it is hoped that the reader will appreciate that the present chapter may be considered a study of the projective aspects of many theorems of euclidean geometry

Ex. 1. *Simson's* theorem for a triangle ABC in E_R is that the feet of the perpendiculars from any point on the circle ABC to the sides of the triangle are in line Obtain the projective aspect of this theorem and then prove it by projective (algebraic) methods.

Ex. 2. *Ceva's* theorem for a triangle ABC in E_R is that if U is any point, not on any side of the triangle, and AU, BU, CU meet BC, CA, AB at D, E, F respectively, then

$$(\overrightarrow{BD}/\overrightarrow{DC}) \,.\, (\overrightarrow{CE}/\overrightarrow{EA}) \,.\, (\overrightarrow{AF}/\overrightarrow{FB}) = 1.$$

If, in E_M, the inaccessible line meets $(BC)'$, $(CA)'$, $(AB)'$ in D_1', E_1', F_1', show that the projective aspect of Ceva's theorem is that

$$\{B, C; D, D_1'\} \,.\, \{C, A; E, E_1'\} \,.\, \{A, B, F, F_1'\} = -1;$$

and show that a similar result is true of any triangle in relation to any point and line in E_M

Ex. 3. *Menelaus'* theorem for a triangle ABC in E_R is that if the sides are met by any transversal, not passing through any

vertex, in points P, Q, R, then

$$(\overrightarrow{BP}/\overrightarrow{PC})\ (\overrightarrow{CQ}/\overrightarrow{QA}) \,.\, (\overrightarrow{AR}/\overrightarrow{RB}) = -1.$$

Obtain, and generalise, the projective aspect of this theorem.

Ex. 4. Obtain, and prove by projective methods, the projective aspect of the theorem that the foot of the perpendicular to a tangent from the focus of a parabola lies on the tangent at the vertex.

(*b*) Some remarks on the use of real diagrams to indicate the behaviour of conics in a complex plane may be useful at this stage.

A conic in the complex plane actually needs a four-dimensional real space for its complete representation by real points. In fact, using complex cartesian co-ordinates x, y, let the equation of the curve be $f(x, y) = 0$. Put

$$x = x' + ix'',\ y = y' + iy'',$$

where x', x'', y', y'' are real. Then the equation takes the form

$$f_1(x', x'', y', y'') + if_2(x', x'', y', y'') = 0,$$

where f_1, f_2 are real quadratic non-homogeneous polynomials Thus the points of the complex curve are represented by the points of the real surface common to the three-dimensional manifolds $f_1 = 0$, $f_2 = 0$ in the real euclidean four-dimensional space in which the co-ordinates are x', x'', y', y''. It is to be emphasised that, although, from this point of view, the complex conic appears to have the dimensions of a surface, yet we still describe it as a curve or ∞^1-locus: the essence of this description is that the points of the complex conic depend algebraically on *one* complex parameter

This representation is incapable of physical realisation. Since experience shows that diagrams of some sort, be they only partly representative, are most useful aids to the mind, we turn to the following considerations.

A conic is determined by five of its points, and, in particular, a conic in the complex plane is determined by its real part if the latter is a real conic (therefore containing an infinity of real points). Thus the real part represents the whole conic in the sense that, from it, the whole conic may be determined

Most of the properties of a conic in which we are interested are projective in character and expressible in terms of relations of incidence. The purpose of a diagram is to exhibit these relations visually. This purpose is achieved if we can find real conics and lines in such relative positions that they show the incidences of interest at the moment.

Thus, for example, the configuration of two conics and their

quadrangle of common points can be illustrated by two ellipses with four real common points: but two real circles would not serve for this purpose. The properties of asymptotes may be shown by using a hyperbola.

It is not always possible to find a perfectly satisfactory real picture: elements which stand in the same relation with regard to a complex conic may have to be represented by elements which stand, in some respects, in different relations with regard to a real conic. For example, if we wish to represent a conic and a self-polar triangle together, one vertex of the real triangle has to be interior to the real conic and the other two have to be exterior. Diagrams which are intended to illustrate properties relative to the circular points have to be drawn so that the circular points are represented by two real accessible points, and the inaccessible line by the line joining them. Such a diagram is, of course, different in appearance from the euclidean figure; and it may be helpful to have two diagrams. For example, the relation of the foci and circular points to a central conic is best exhibited by the figure of an ellipse inscribed in a quadrilateral, two of whose opposite vertices represent the circular points, the other four representing the four foci: but the property, say, that the feet of the perpendiculars from the foci to a tangent lie on the auxiliary circle is most convincingly shown by the figure of the real euclidean case.

CHAPTER V

FURTHER PROPERTIES OF CONICS

39. The harmonic conic of two conic-envelopes.

(*a*) We are concerned here with a generalisation, in a modified complex euclidean plane, of the notion of director-circle of a conic. The director-circle is the locus of the intersection of a pair of perpendicular lines belonging to the associated conic-envelope. It may therefore be described as the locus of the intersection of a pair of lines of the conic-envelope which are conjugate with regard to the conic-envelope (I, J), or as the locus of the intersection of a pair of lines through I, J conjugate with regard to the given conic-envelope.

(*b*) The last two forms of the description lead to obvious generalisations. First of all we may replace I, J by any other pair of points. Then we find, without any difficulty, that the equation (in any system of homogeneous co-ordinates) of the locus of the intersection of conjugate lines, one passing through each of the fixed points (x_1, y_1, z_1), (x_2, y_2, z_2), with regard to the conic $S = 0$, where

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

is the conic

$$SS_{12} = S_1S_2,$$

in the usual notation, passing through the two fixed points. This conic is called the *director-conic* of $S = 0$ relative to the two fixed points.

Ex. 1. If the conic-envelope associated with $S = 0$ has the equation $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$ and if the equation of the reducible conic-envelope consisting of the pencils of lines with vertices at the two fixed points has the equation $A'l^2 + B'm^2 + C'n^2 + 2F'mn + 2G'nl + 2H'lm = 0$, verify that the equation of the director-conic takes the form

$$\begin{aligned}&(BC' + B'C - 2FF')x^2 + 2(GH' + G'H - AF' - A'F)yz\\ &+ (CA' + C'A - 2GG')y^2 + 2(HF' + H'F - BG' - B'G)zx\\ &+ (AB' + A'B - 2HH')z^2 + 2(FG' + F'G - CH' - C'H)xy = 0.\end{aligned}$$

(*c*) A further generalisation is to replace the reducible conic-envelope, consisting of the pencils with vertices at the two fixed points, by an irreducible conic-envelope. We therefore now seek

the locus of a point such that the lines of the conic envelope $\Sigma = 0$ which pass through the point are harmonic with respect to the lines of the conic-envelope $\Sigma' = 0$ which pass through the point, where

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm,$$
$$\Sigma' \equiv A'l^2 + B'm^2 + C'n^2 + 2F'mn + 2G'nl + 2H'lm.$$

We prove first that $\lambda_1 + \lambda_2 = 0$, where λ_1, λ_2 are the parameters of the two conic-envelopes in the range $\Sigma + \lambda\Sigma' = 0$ which pass through such a point. This is equivalent to $\{0, \infty ; \lambda_1, \lambda_2\} = -1$; 0 and ∞ are the parameters of $\Sigma = 0$, $\Sigma' = 0$ respectively.

Let (l_1, m_1, n_1), (l_2, m_2, n_2) be two lines through the point. Then any line through the point has co-ordinates of the form $(l_1 + \theta l_2, m_1 + \theta m_2, n_1 + \theta n_2)$. The θ-parameters of those lines which belong to $\Sigma = 0$ are θ_1, θ_2, the roots of

$$\Sigma_{11} + 2\theta\Sigma_{12} + \theta^2\Sigma_{22} = 0;$$

and the θ-parameters of those lines which belong to $\Sigma' = 0$ are θ_3, θ_4, the roots of

$$\Sigma_{11}' + 2\theta\Sigma_{12}' + \theta^2\Sigma_{22}' = 0.$$

The hypothesis is that $\{\theta_1, \theta_2;\ \theta_3, \theta_4\} = -1$, which is the same as $(\theta_1 + \theta_2)(\theta_3 + \theta_4) = 2(\theta_1\theta_2 + \theta_3\theta_4)$, for this it is necessary and sufficient that

$$2\Sigma_{12}\Sigma_{12}' = \Sigma_{11}\Sigma_{22}' + \Sigma_{11}'\Sigma_{22}.$$

The parameters λ_1, λ_2 are the values of λ which make the quadratic in θ

$$(\Sigma_{11} + \lambda\Sigma_{11}') + 2\theta(\Sigma_{12} + \lambda\Sigma_{12}') + \theta^2(\Sigma_{22} + \lambda\Sigma_{22}') = 0$$

have equal roots and are therefore given by

$$(\Sigma_{12} + \lambda\Sigma_{12}')^2 = (\Sigma_{11} + \lambda\Sigma_{11}')(\Sigma_{22} + \lambda\Sigma_{22}').$$

By the relation just before proved, it follows that $\lambda_1 + \lambda_2 = 0$.

The point equation of the conic associated with the conic-envelope $\Sigma + \lambda\Sigma' = 0$ is

$$\Delta S + \lambda K + \lambda^2\Delta' S' = 0,$$

where Δ, Δ', S, S' have their usual significances and

$$\begin{aligned} K \equiv (BC' + B'C - 2FF')x^2 &+ 2(GH' + G'H - AF' - A'F)yz \\ + (CA' + C'A - 2GG')y^2 &+ 2(HF' + H'F - BG' - B'G)zx \\ + (AB' + A'B - 2HH')z^2 &+ 2(FG' + F'G - CH' - C'H)xy. \end{aligned}$$

Therefore, if (x_0, y_0, z_0) are the co-ordinates of the point already in question, λ_1, λ_2 are also the roots of

$$\Delta S_{00} + \lambda K_{00} + \lambda^2\Delta' S'_{00} = 0.$$

Therefore $K_{00} = 0$. The locus of the point is therefore the conic $K = 0$; this is called the *harmonic conic* of the two given conic-envelopes.

Ex. 2. The harmonic conic passes through the eight contact-points of the four lines common to the two conic-envelopes.

This may be restated in the form: two conics touch the sides of a quadrilateral, the eight points of contact lie on a conic.

Ex 3. The circle having two given points A, B as the extremities of a diameter is the harmonic conic of the conic-envelopes (A, B) and (I, J).

40. The harmonic conic-envelope of two conics.

(*a*) The lines which meet two conics $S = 0$, $S' = 0$ in pairs of points which are harmonic with respect to each other generate a conic-envelope, called the *harmonic conic-envelope* of the given conics.

By dualising the algebra of the last part of section 39, the equation of this conic-envelope is seen to be $\Phi = 0$, where

$$\begin{aligned}\Phi \equiv \quad & (bc' + b'c - 2ff')l^2 + 2(gh' + g'h - af' - a'f)mn \\ & + (ca' + c'a - 2gg')m^2 + 2(hf' + h'f - bg' - b'g)nl \\ & + (ab' + a'b - 2hh')n^2 + 2(fg' + f'g - ch' - c'h)lm.\end{aligned}$$

Ex. 1. The eight tangents to $S = 0$, $S' = 0$ at the common points of these conics belong to $\Phi = 0$.

Ex 2. The harmonic conic-envelope of the asymptotes and director-circle of a non-parabolic conic is the conic-envelope associated with this conic.

(*b*) With modified complex cartesian co-ordinates, the harmonic conic-envelope of the circles

$$\begin{aligned}x^2 + y^2 - 2uxz - 2vyz + cz^2 &= 0,\\ x^2 + y^2 - 2u'xz - 2v'yz + c'z^2 &= 0,\end{aligned}$$

is

$$\begin{aligned}(c + c' - 2uu' - 2vv')(l^2 + m^2)& \\ + 2(ul + vm + n)(u'l + v'm + n) &= 0.\end{aligned}$$

This conic-envelope contains the four lines joining the centres of the circles, namely $(u, v, 1)$ and $(u', v', 1)$, to the circular points; the centres of the circles are therefore corresponding foci of the conic-envelope.

If $c + c' - 2uu' - 2vv' = 0$, the situation arises that every line through the centre of one circle meets the other circle in two points conjugate in regard to the first circle; and, in particular, the tangents from the first centre to the second circle must have their points of contact on the first circle As is trivial to prove,

the tangents to the first circle at these points are perpendicular to the radii to the points; therefore at each accessible intersection the tangents to the two circles are perpendicular. The circles are then said to intersect *orthogonally*.

That $c + c' - 2uu' - 2vv' = 0$ is a necessary and sufficient condition for the circles to intersect orthogonally may be seen as follows.

The tangents at a common accessible point (x_1, y_1, z_1) are

$$\begin{aligned} x(x_1 - uz_1) + y(y_1 - vz_1) + z(-ux_1 - vy_1 + cz_1) &= 0 \\ x(x_1 - u'z_1) + y(y_1 - v'z_1) + z(-u'x_1 - v'y_1 + c'z_1) &= 0. \end{aligned}$$

These are perpendicular if and only if

$$(x_1 - uz_1)(x_1 - u'z_1) + (y_1 - vz_1)(y_1 - v'z_1) = 0.$$

Since also

$$\begin{aligned} x_1^2 + y_1^2 - 2ux_1z_1 - 2vy_1z_1 + cz_1^2 &= 0, \\ x_1^2 + y_1^2 - 2u'x_1z_1 - 2v'y_1z_1 + c'z_1^2 &= 0, \end{aligned}$$

an equivalent condition is

$$z_1^2(c + c' - 2uu' - 2vv') = 0,$$

or, since $z_1 \neq 0$,

$$c + c' - 2uu' - 2vv' = 0.$$

It will be noticed that this proof shows that if the tangents at one common accessible point are perpendicular, so are the tangents at the other such point.

41. Coaxal circles.

In a modified complex euclidean plane, the pencil of circles determined by two given circles, which do not touch at either of the circular points I, J, is called a *coaxal system* of circles. The circles of the system have in common two accessible points A, B which may coincide, in which case the circles have a common tangent at A.

(i) The general case.—We suppose that A, B are distinct. The system contains three pairs of lines as members. One consists of the inaccessible line IJ together with AB; AB is called the *radical axis* of the system. The other two pairs of lines are AI, BJ, meeting at F, and AJ, BI, meeting at G; these are point-circles; their centres F, G are called the *limiting points* of the system (Figs. 81, 82, 83).

Let E be the common point of AB, IJ. The triangle EFG is self-polar with regard to every circle of the system. The centre of every circle, being the pole of IJ which passes through E,

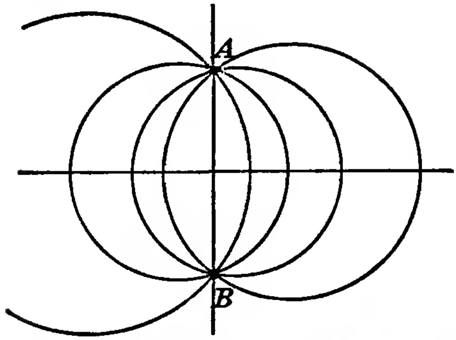


FIG. 81.—REAL COAXAL CIRCLES WITH REAL ACCESSIBLE COMMON POINTS.

therefore lies on FG, and, conversely, every point on FG is the centre of one circle of the system. FG is called the *line of centres*

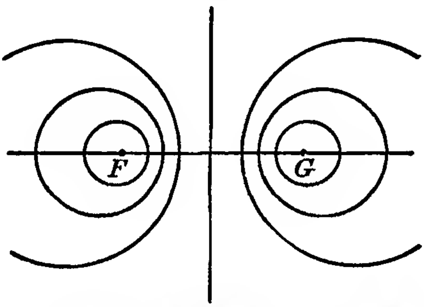


FIG. 82.—REAL COAXAL CIRCLES WITH REAL LIMITING POINTS.

of the system. It is a harmonic property of the quadrangle $ABIJ$ that FG, FE *harm* FI, FJ; the radical axis, being parallel to FE, is therefore perpendicular to the line of centres.

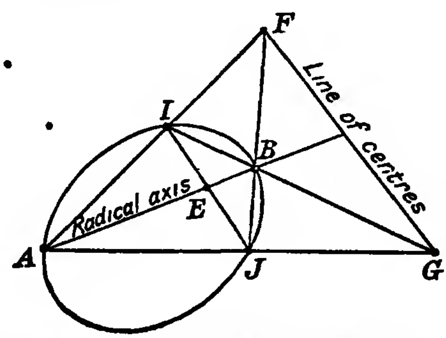


FIG. 83.—THE GENERAL FIGURE FOR A SYSTEM OF COAXAL CIRCLES.

Every line through E meets any circle of the system in two points whose mid point is on FG; and such lines are perpendicular to FG. The figure of the system of circles is therefore symmetrical about the line of centres. From this it follows that any common tangent of two circles of the system meets the line of centres in a point from which can be drawn a second common tangent.

Associated with the co-axal system intersecting in A, B is a second coaxal system consisting of the circles intersecting in F, G. The limiting points of the second system are A, B We prove that any circle of the first system is orthogonal to any circle of the second system.

A triangle of reference, symmetrically related to both coaxal systems, is IJK, where $K = AB\,.\,FG$. Take A as unit point, then if JK is $x = 0$, KI is $y = 0$ and IJ is $z = 0$, the points F, G, B have co-ordinates $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ respectively.

Every circle through A, B has an equation of the form

$$\lambda(z^2 - xy) + z(x - y) = 0,$$

and its centre is the point $(-1, 1, \lambda)$; and every circle through F, G has an equation of the form

$$\mu(z^2 + xy) + z(x + y) = 0,$$

its centre being the point $(-1, -1, \mu)$.

The harmonic conic-envelope of these two circles has the equation

$$l^2 - m^2 + \lambda\mu n^2 + (\mu - \lambda)mn - (\lambda + \mu)nl = 0,$$

which may be rewritten as

$$(-l + m + \lambda n)(-l - m + \mu n) = 0.$$

The harmonic conic-envelope therefore consists of the pencils of lines with vertices at the centres of the two circles; therefore the two circles cut orthogonally.

Ex. 1. With modified co-ordinates based on a common distance-scale, the coaxal system associated with the system $x^2 + y^2 + 2\lambda xz + cz^2 = 0$ is $x^2 + y^2 + 2\mu yz - cz^2 = 0$.

Ex 2. By the *radical axis of two circles* we mean the radical axis of the coaxal system determined by these circles.

Prove that the radical axes derived from three circles by taking them in pairs are concurrent; and show that the point of concurrence is the centre of a circle orthogonal to all three given circles.

(ii) Special case.—*We consider now the case of a coaxal system of circles determined by two circles which touch at an*

accessible point A. Let the common tangent of the two circles meet the inaccessible line at E. Then the coaxal system is the pencil of conics passing through I, J and touching EA at A. EA is called the *radical axis* of the system.

Let $K \equiv (I, J)/E$, then AK is the polar of E with regard to every circle of the system and therefore contains the centre of every such circle. AK is called the *line of centres*; it is perpendicular to the radical axis.

Exactly as in part (i), we prove that the figure of the system of circles is symmetrical about the line of centres and that the common tangents of any two of the circles meet by pairs on the line of centres.

The associated coaxal system consists of the circles touching KA at A. Every circle of this system meets every circle of the first system orthogonally at A and therefore also at the remaining accessible intersection.

Ex. 3. Three circles touch in pairs at A, B, C. Prove that the tangents at A, B, C meet in the centre of the circle through A, B, C and that this circle is orthogonal to all three given circles.

(iii) **Note.**—The real circles of a coaxal system determined by two real circles cover a system of circles in the real euclidean plane embedded in the modified complex plane. The circles in the real plane are called a coaxal system in that plane; properties of this system may be inferred immediately from what has been said about coaxal systems in the complex plane

In particular, it is to be remarked that if the circles of a coaxal system in a real plane meet in distinct real points A, B then there are no real limiting points (in the covering complex plane, F, G are conjugate unreal points); and if the system has real limiting points F, G then there are no real intersections (in the covering complex plane, A, B are now conjugate unreal points).

The fact that the circles of two coaxal systems in a real plane form an orthogonal network of curves has important applications to various branches of mathematics.

42. Confocal conics.

(i) **The general case.**—Let $\Sigma = 0$ be the equation of an irreducible conic-envelope which does not contain the inaccessible line Every conic-envelope having the same foci as $\Sigma = 0$ contains the four lines common to $\Sigma = 0$ and to the conic-envelope (I, J), say $\Sigma' = 0$, and the converse statement is true These conic-envelopes, with their associated conics, are said to form a *confocal system* (Fig. 84) The equation of the system is $\Sigma + \lambda\Sigma' = 0$.

The equation of the system takes a simple form if we choose

the triangle of reference and the unit point as follows. Let F, F' and H, H' be the two pairs of opposite foci, with F, H, I in line. Take HH' to be $x = 0$, FF' to be $y = 0$, and IJ to be $z = 0$.

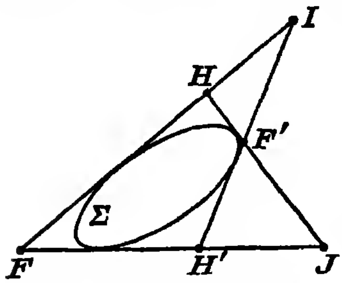


Fig. 84.—Confocal Conics

The unit point is fixed by assigning to I a triad of co-ordinates $(1, i, 0)$.

The triangle of reference is the diagonal triangle of the quadrilateral of lines common to all the conic-envelopes; therefore J has co-ordinates $(1, -i, 0)$. And, the triangle of reference being a self-polar triangle relative to every conic-envelope of the range, Σ has the form $al^2 + bm^2 - n^2$.

The equation of the confocal system may therefore be put in the form

$$(al^2 + bm^2 - n^2) + \lambda(l^2 + m^2) = 0$$

or

$$(a + \lambda)l^2 + (b + \lambda)m^2 = n^2,$$

to which corresponds the point-equation

$$\frac{x^2}{a + \lambda} + \frac{y^2}{b + \lambda} = z^2.$$

Ex. 1. The foci F, F' have co-ordinates $(\pm(a - b)^{\frac{1}{2}}, 0, 1)$ and the foci H, H' have co-ordinates $(0, \pm(b - a)^{\frac{1}{2}}, 1)$.

Ex. 2. The conic associated with the conic-envelope

$$\Sigma + \lambda(l^2 + m^2) = 0,$$

where

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm,$$

has, in the usual notation, the equation

$$\Delta S + \lambda K + \lambda^2 z^2 = 0,$$

where

$$K \equiv C(x^2 + y^2) + (A + B)z^2 - 2Fyz - 2Gzx.$$

$K = 0$ is the director-circle of the conic $S = 0$.

Ex. 3. The co-ordinates being modified complex, prove that the equation of the confocal system having the accessible points (x_1, y_1, z_1), (x_2, y_2, z_2) as a pair of opposite foci is

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2) + \lambda(l^2 + m^2) = 0.$$

(ii) A property of orthogonality.—It has been remarked earlier that the locus of the poles of a given line P in regard to a range of conic-envelopes is another line Q. There is one point of Q, in

the case of a confocal system, arising from the conic-envelope (I, J); this point is the harmonic conjugate, relative to I, J, of the intersection of P with IJ; therefore Q is perpendicular to P. And there is one conic-envelope of the system containing P; the contact point of P is P . Q. Hence Q is the normal at the contact point.

Algebraically, the pole of P (l_1, m_1, n_1) with regard to the conic whose parameter is λ has co-ordinates $(l_1(a + \lambda), m_1(b + \lambda), - n_1)$. Its locus is therefore the line Q $(m_1 n_1, \ - n_1 l_1, \ (a - b) l_1 m_1)$, which is perpendicular to P. From this it follows at once that P is the locus of the poles of Q.

Consider now the two conic-envelopes of the system containing P, Q respectively. The associated conics intersect at P . Q, and therefore intersect orthogonally at the point. And, since both conics are symmetrical about the common axes FF', HH', they intersect orthogonally at each of their remaining common points.

There are, in fact, through every point (x_1, y_1, z_1), not on a common line of the system, two of the associated conics, whose parameters λ_1, λ_2 are the roots of the quadratic in λ

$$\frac{x_1^2}{a + \lambda} + \frac{y_1^2}{b + \lambda} = z_1^2.$$

Inserting the two values for λ in this equality and subtracting, we have

$$\frac{x_1^2}{(a + \lambda_1)(a + \lambda_2)} + \frac{y_1^2}{(b + \lambda_1)(b + \lambda_2)} = 0,$$

which is a necessary and sufficient condition that the tangents to the two conics at (x_1, y_1, z_1) should be perpendicular.

Ex. 4. Let P be a variable line through the point U (x_0, y_0, z_0), not on any side of the fundamental triangle. Show that the line Q, referred to in the text above, generates a parabolic conic-envelope; and find the focus V of this. Show that, if P now turns about V instead of U, then Q generates a parabolic conic-envelope with focus at U. Show further that OU, OV make equal angles with the axes of the confocal system, O being the point (0, 0, 1).

(iii) **An interval property.**—The pairs of tangents from a point P to the conics of a confocal system (that is, the pairs of lines through P belonging to the conic-envelopes of the system) form an involution whose double lines are the perpendicular lines P, Q which touch at P the two confocal conics through the point. Being perpendicular, these lines bisect the intervals between the tangents from P to any one of the conics and, in particular, bisect the intervals between PF, PF', where F, F' are opposite foci.

Ex. 5. The locus of the point in which a tangent to the conic $x^2/a + y^2/b = z^2$ meets a perpendicular tangent to the confocal conic $x^2/(a + \lambda) + y^2/(b + \lambda) = z^2$ is the circle $x^2 + y^2 = (a + b + \lambda)z^2$, the co-ordinate system being as in part (i). Interpret this statement when $\lambda = 0$ and when $\lambda = -b$.

Ex. 6. In regard to the real euclidean plane embedded in the complex plane, the system of conics covered by a confocal system may also be called confocal.

Of the conics confocal with the ellipse $x^2/a^2 + y^2/b^2 = 1$ which pass through a given point, one is an ellipse and the other a hyperbola. There is thus set up, in the real plane, a network of ellipses and hyperbolas, such that every ellipse meets every hyperbola orthogonally. Show that each ellipse is either

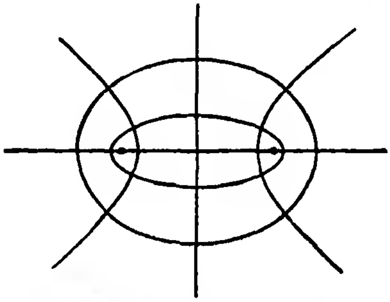


FIG 85—REAL CONFOCAL CONICS.

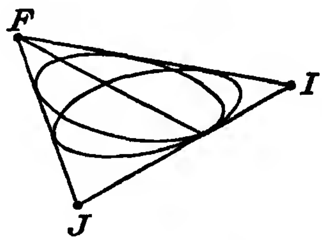


FIG 86—CONFOCAL PARABOLAS.

wholly interior or wholly exterior to every confocal ellipse; and that a similar remark applies to each hyperbola (Fig 85).

Ex. 7. The harmonic conic of the two confocal conic-envelopes

$$al^2 + bm^2 = n^2,\ (a + \lambda)l^2 + (b + \lambda)m^2 = n^2$$

may have its equation expressed in either of the forms

$$2ab\left(\frac{x^2}{a} + \frac{y^2}{b} - z^2\right) + \lambda(x^2 + y^2 - (a + b)z^2) = 0,$$

$$2(a + \lambda)(b + \lambda)\left(\frac{x^2}{a + \lambda} + \frac{y^2}{b + \lambda} - z^2\right) - \lambda(x^2 + y^2 - (a + b + 2\lambda)z^2) = 0$$

Thus a conic passes through the eight points where each of the given conics is met by its director-circle. This result is contained in *Ex.* 2 of section 39.

Ex. 8. The parameters λ_1, λ_2 of the conics of the system $x^2/(a + \lambda) + y^2/(b + \lambda) = z^2$ which pass through the point (x_1, y_1, z_1) are such that

$$x_1^2 : y_1^2 : z_1^2 = \frac{(a + \lambda_1)(a + \lambda_2)}{a - b} : \frac{(b + \lambda_1)(b + \lambda_2)}{b - a} : 1.$$

To each point (x_1, y_1, z_1) corresponds in general one unordered pair of numbers λ_1, λ_2; to each unordered pair of numbers λ_1, λ_2 correspond in general four points (x_1, y_1, z_1), namely the intersections of two confocal conics. In the case of a confocal system of ellipses and hyperbolas in a real plane, λ_1, λ_2 are called *confocal co-ordinates* of the point (x_1, y_1, z_1).

(iv) Confocal parabolas.—Now let us suppose that the irreducible conic-envelope $\Sigma = 0$ contains the inaccessible line, the contact point being C; and let F be the focus. Then if $\Sigma' = 0$ is the equation of the conic-envelope (I, J), the conic-envelopes of the range $\Sigma + \lambda\Sigma' = 0$ are parabolic and all have the same focus F and axis FC, the range together with its associated conics is called a *confocal system* of parabolas (Fig. 86).

There are two triangles each of which it is natural to consider as a triangle of reference. First, we may take $x = 0$ to be FJ, $y = 0$ to be FI, $z = 0$ to be IJ; then let us take the unit point on FC by assigning to FC the equation $x - y = 0$. The equations of the conic-envelopes (F, C) and (I, J) are respectively

$$(l + m)n = 0, lm = 0;$$

hence the equation of the range takes the form

$$mn + nl + \lambda lm = 0,$$

corresponding to the point equation

$$x^2 + y^2 + \lambda^2 z^2 = 2\lambda yz + 2\lambda zx + 2xy.$$

Next, we may take IJ to be $z = 0$, FC to be $y = 0$, and the harmonic conjugate of FC with regard to FI, FJ (that is, the common latus rectum) as $x = 0$. And we may so choose the unit point that I has co-ordinates $(1, i, 0)$; J then has co-ordinates $(1, -i, 0)$. The conic-envelopes (F, C) and (I, J) now have equations

$$nl = 0, l^2 + m^2 = 0$$

respectively; and therefore the equation of the range is

$$2nl + \lambda(l^2 + m^2) = 0,$$

corresponding to the point equation

$$y^2 = \lambda^2 z^2 - 2\lambda zx.$$

Ex. 9. Through every accessible point there pass two parabolas of the system and these intersect orthogonally.

43. Cases of a theorem of Poncelet.

In section 22 (iii) (f), we proved the simplest case of a theorem due to *Poncelet* (see section 59 (ii)), namely that:

If one triangle can be constructed to have its vertices on one conic and its sides touching another conic, then an infinity of such triangles can be constructed for the two conics.

Another case of Poncelet's theorem is now given as an application of the concept of director-conic. Metrically, the theorem to be proved amounts to the easily proved statement that an infinity of rectangles can be constructed so that their sides touch a given conic and their vertices lie on the director-circle of the conic.

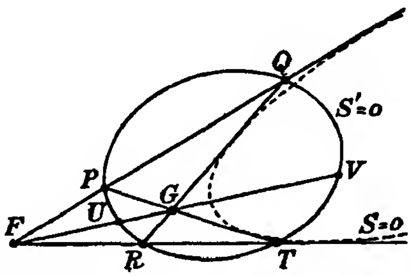


FIG. 87.—FIGURE RELATING TO A SYSTEM OF QUADRANGLES.

Let $S = 0$, $S' = 0$ be two given conics such that there exists a quadrangle having its vertices P, Q, R, T on $S' = 0$ and its sides PQ, QR, RT, TP touching $S = 0$ (Fig. 87). We prove that an infinity of such quadrangles can be constructed.

Let PQ meet RT at F and let QR meet TP at G; and let $S' = 0$ meet FG at U, V. The director-conic of $S = 0$ relative to U, V passes through the six points P, Q, R, T, U, V and is therefore the same as $S' = 0$, with which it has all these points in common.

Therefore, starting with any two points F', G' on UV, which are harmonic with respect to U, V, and constructing the tangents from these points to $S = 0$, we obtain a figure of four points and four lines having the required property. Clearly, every point of $S' = 0$ and every tangent of $S = 0$ belongs to just *one* such figure.

Ex. 1. Let A be the pole of UV with regard to $S = 0$ and let B, C be the two points on FG, harmonic with regard to both F, G and U, V. By taking ABC as triangle of reference and assigning to U co-ordinates $(1, i, 0)$, show that S, S' take the forms $bx^2 + ay^2 - abz^2$ and $x^2 + y^2 - (a + b)z^2$ respectively.

44. Reciprocation.

(*a*) Let $S = 0$ be the equation of an irreducible conic, where

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

The co-ordinates (x_1, y_1, z_1) of a point P and the co-ordinates (l_1, m_1, n_1) of its polar line P are then connected by the equations

$$l_1 : m_1 : n_1 = ax_1 + hy_1 + gz_1 : hx_1 + by_1 + fz_1 : gx_1 + fy_1 + cz_1,$$

which are equivalent, in the usual notation, to

$$x_1 : y_1 : z_1 \\ = Al_1 + Hm_1 + Gn_1 : Hl_1 + Bm_1 + Fn_1 : Gl_1 + Fm_1 + Cn_1.$$

These equations, connecting the co-ordinates of a point and of a line, belong to a class, called *correlations*, which we shall consider in detail later.

The symmetry and linearity of the equations lead at once to the proposition that, if P describes a line Q, then P turns about the pole Q of Q; and P corresponds to P in a projectivity between the points of Q and the lines through Q. The proof, which is trivial, depends on the fact that, if $P \equiv \lambda P_1 + \mu P_2$, then $\mathsf{P} \equiv \lambda \mathsf{P}_1 + \mu \mathsf{P}_2$.

Next, we observe that if P describes a conic $S' = 0$, where

$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy,$$

then P generates the conic-envelope $\Sigma'' = 0$, where

$$\Sigma'' \equiv a'(Al + Hm + Gn)^2 + \dots \\ + 2f'(Hl + Bm + Fn)(Gl + Fm + Cn) + \dots;$$

and, if P generates a conic-envelope, P describes a conic.

Generally, let Ψ be a figure of points and lines and let Ψ' be the figure of the polars of these points and the poles of these lines with regard to $S = 0$. The Ψ' is said to be the *reciprocal* of Ψ relative to the conic $S = 0$. The relation is reflexive: Ψ is the reciprocal of Ψ'. Moreover, as is evident, the figure Ψ' is dual to the figure Ψ and harmonic relations between the elements of Ψ correspond to harmonic relations between the reciprocal elements of Ψ'; and *vice versa.*

Ex. 1. If Ψ consists of a conic and an inscribed triangle, then Ψ' consists of a conic-envelope and a circumscribed triangle.

Ex. 2. The reciprocal of a conic and a triangle self-polar with regard to it is a conic-envelope and a triangle self-polar with regard to this.

(*b*) For convenience we say that, if the conic-envelope $\Sigma'' = 0$ is reciprocal to the conic $S' = 0$, then the conic $S'' = 0$, associated with the conic-envelope, is reciprocal to the conic $S' = 0$; and a similar convention applies to the conic-envelope $\Sigma' = 0$.

Any two irreducible conics $S' = 0$, $S'' = 0$ are reciprocal with regard to at least one other conic $S = 0$. The proof of this theorem is now outlined and, for brevity, the conics are referred to as S', S'', S.

A point having the same polar line for S' as for S'' must reciprocate, with regard to such a conic as S, into a line having the same pole for S' as for S''. And the common points of S', S'' must reciprocate into the common tangents of S', S''.

A point P having the same polar Q for S' as for S must also be the pole of Q for S''. Thus if S', S'' have a unique common self-polar triangle, this is the only common self-polar triangle of S with either S' or S''.

If S', S'' touch at one point only, S must touch them at this point, and any two of S, S', S'' must have the same order of contact.

If S', S'' touch at two points, S must either touch them at these two points or else the triangle formed by the common tangents and the chord of contact must be self-polar relative to S.

Thus the following cases occur:

(α) When S', S'' have a unique common self-polar triangle there are four conics S, each having this triangle self-polar Referring co-ordinates to this triangle, we can take

$$S' \equiv a'x^2 + b'y^2 + c'z^2, \quad S'' \equiv a''x^2 + b''y^2 + c''z^2,$$

and then the four conics S are given by

$$S \equiv ax^2 + by^2 + cz^2 \quad \text{with } a^2 = a'a'', b^2 = b'b'', c^2 = c'c''$$

(β) When S', S'' touch at one point O and meet at two other points, there are two conics S, each touching S', S'' at O. We can take

$$S' \equiv x^2 + y^2 + 2yz, \quad S'' \equiv x^2 + y^2 + 2\lambda yz,$$

and then

$$S \equiv x^2 + by^2 + 2fyz \quad \text{with } b = \tfrac{1}{2}(f + 1/f), f^2 = \lambda.$$

(γ) When S', S'' have three-point contact, we can take

$$S' \equiv zx - y^2, \quad S'' \equiv zx - y^2 + 2\lambda yz,$$

and then there is just one conic S given by

$$S \equiv zx - y^2 + \lambda yz + \tfrac{1}{8}\lambda^2 z^2.$$

(δ) When S', S'' have four-point contact, we can take

$$S' \equiv zx - y^2, \quad S'' \equiv zx - y^2 + \lambda z^2$$

and then either

$$S \equiv zx - y^2 + \tfrac{1}{2}\lambda z^2$$

or, with any f,

$$S \equiv zx + y^2 + 2fyz + \tfrac{1}{2}(f^2 + \lambda)z^2.$$

(ϵ) When S', S'' touch at two points A, B, there are two conics S which touch them at A, B, and also two conics S for which any given common self-polar triangle of S', S'' is also self-polar. We can take

$$S' \equiv zx - y^2, \quad S'' \equiv zx - \lambda y^2,$$

and then either

$$S \equiv zx + \theta y^2 \quad \text{with } \theta^2 = -\lambda,$$

or else

$$S \equiv ax^2 + by^2 + cz^2 \quad \text{with } b^2 = 4ca\lambda^2.$$

(*c*) Reciprocation has interesting metrical properties when it is carried out relative to a circle.

The reciprocal of a circle relative to another circle is, in general, a conic with one focus at the centre of the second circle, and conversely; but, if the two circles are concentric, the reciprocal is a third concentric circle. The proof is immediate.

Let us now consider a confocal system of conics; let F, F' be a pair of opposite foci and let I, J be the circular points. We reciprocate relative to any circle with centre at F.

The confocal conics touch FI, FJ, therefore their reciprocals pass through I, J and are circles Moreover, the conics all touch the accessible lines $F'I$, $F'J$; the circles therefore all pass through the poles A, B of these lines relative to the circle of reciprocation. The reciprocal of the confocal system of conics is thus a co-axal system of circles The radical axis is the reciprocal of F' and the limiting points are the reciprocals of IJ, namely F, and of the line joining the other pair of opposite foci.

Ex. 3. Show that the reciprocal of a system of coaxal circles relative to any circle with centre at a limiting point is a system of confocal conics What is the reciprocal of the orthogonal system of coaxal circles?

Ex. 4 Prove the theorem of section 33 (x) by reciprocation.

Ex. 5. A variable conic, for which a fixed point F and a fixed line D are respectively focus and corresponding directrix, meets another fixed line L at P, Q. Prove that the tangents at P, Q generate a conic-envelope which contains D, L and has a focus at F.

Ex. 6. The sides of a triangle PQR touch a parabola whose focus is F. Prove that the perpendiculars through P, Q, R to FP, FQ, FR respectively are concurrent.

Ex 7. Prove that the reciprocal of a parabola relative to a circle whose centre is on the directrix is a rectangular hyperbola.

Ex 8. O is a fixed point on a conic and P, Q are variable points on the conic such that OP is perpendicular to OQ. Prove, by reciprocation, that PQ passes through a fixed point.

Ex. 9. The directrices of the parabolas having a given focus and touching a given line pass through the image point of the focus in this line

45. Outpolar and inpolar conics.

(i) Outpolarity and inpolarity.—We consider now in detail the relation between a conic $S = 0$ and a conic-envelope $\Sigma' = 0$, when

it is possible to inscribe a triangle in $S = 0$ so as to be self-polar with regard to $\Sigma' = 0$. In these circumstances we say that $S = 0$ is *outpolar* to $\Sigma' = 0$. It may be stated now, though it is convenient to defer the proof until we have initiated the theory of the invariants of pairs of conics under linear transformations of the co-ordinates, that a necessary and sufficient condition for the conic to be outpolar to the conic-envelope is $\Theta' = 0$, where, in the usual notation,

$$\Theta' \equiv aA' + bB' + cC' + 2fF' + 2gG' + 2hH'.$$

The importance of this condition lies essentially in its linearity in the coefficients of each of S, Σ'.

There is a dual relation between a conic-envelope $\Sigma = 0$ and a conic $S' = 0$ when it is possible to find a triangle whose sides belong to $\Sigma = 0$ and which is self-polar with regard to $S' = 0$. We then say that $\Sigma = 0$ is *inpolar* to $S' = 0$; and a necessary and sufficient condition for this is $\Theta = 0$, where

$$\Theta \equiv Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh'.$$

The algebraic conditions show at once that, if $S = 0$ is outpolar to $\Sigma' = 0$, then $\Sigma' = 0$ is inpolar to $S = 0$. The proof of this fact, in the case where $S = 0$, $\Sigma' = 0$ are both irreducible, is also immediate by reciprocation, for then, as we showed in the previous section, there exists an irreducible conic with regard to which the polars of the vertices of the triangle, inscribed in $S = 0$ and self-polar with regard to $\Sigma' = 0$, are three lines which belong to $\Sigma' = 0$ and form a triangle self-polar with regard to $S = 0$.

If $S = 0$ is a pair of different lines P, Q and outpolar to an irreducible conic-envelope $\Sigma' = 0$, it is evident from the definition that P, Q are conjugate with regard to $\Sigma' = 0$; and the condition $\Theta' = 0$ is easily verified Then the two lines of $\Sigma' = 0$ which pass through the point $\mathsf{P} . \mathsf{Q}$, together with any other line of $\Sigma' = 0$, form a triangle self-polar with regard to $S = 0$, according to the usual conventions.

If $S = 0$ is irreducible and outpolar to a reducible conic-envelope $\Sigma' = 0$, consisting of the lines through two different points P, Q, a triangle self-polar with regard to $\Sigma' = 0$ must necessarily have PQ as one side and its remaining sides harmonic with regard to P, Q. When such a triangle is inscribed in $S = 0$, it follows that P, Q are conjugate with regard to the conic. The condition $\Theta' = 0$ is again easily verified.

Finally, if $S = 0$ consists of two lines P, Q and $\Sigma' = 0$ consists of the pencils with vertices at two points P, Q, the conic being outpolar with regard to the conic-envelope, then a triangle inscribed in $S = 0$ and self-polar with regard to $\Sigma' = 0$ must be formed by PQ, P (or Q) and some line through $\mathsf{Q} . PQ$ (or $\mathsf{P} . PQ$). Therefore P, Q are harmonic with regard to P, Q; and a triangle

belonging to $\Sigma' = 0$ and self-polar with regard to $S = 0$ consists of PQ, PR, QR, where $R = P \cdot Q$.

The reader is left to consider the possibilities arising when $S = 0$ consists of a line counted twice or $\Sigma' = 0$ consists of a pencil counted twice.

(ii) A theorem connected with outpolarity.—We have remarked in other words in section 22, and indicated an algebraic proof of the fact, that if $S = 0$ is outpolar to $\Sigma' = 0$, then an infinity of triangles can be inscribed in $S = 0$ each to be self-polar with regard to $\Sigma' = 0$. Then, also, it follows that there exist an infinity of triangles belonging to $\Sigma' = 0$ which are each self-polar with regard to $S = 0$.

A geometrical proof of the first theorem may be of interest; the second theorem follows by duality. In view of the remarks already made in part (i) of this section, we restrict ourselves to the case where both $S = 0$ and $\Sigma' = 0$ are irreducible.

First, let ABC, $A'B'C'$ be two triangles self-polar with regard to $\Sigma' = 0$ and have no vertex of either on any side of the other. Then, if $P' = A'B' \cdot BC$ and $Q' = A'C' \cdot BC$, the poles of AB, AC, AB', AC' are respectively C, B, Q', P' and the four lines correspond with the four points, in the orders named, in a projectivity. Therefore

$$A(BCB'C') \,\overline{\wedge}\, (CBQ'P') \,\overline{\wedge}\, (BCP'Q') \,\overline{\wedge}\, A'(BCP'Q') \,\overline{\wedge}\, A'(BCB'C').$$

Consequently A, A', B, C, B', C' lie on a conic.

Next, let us suppose that UVW is the triangle, inscribed in $S = 0$ and self-polar relative to $\Sigma' = 0$, in virtue of which $S = 0$ is outpolar to $\Sigma' = 0$ Take any point U' on $S = 0$ and let its polar with regard to $\Sigma' = 0$ meet $S = 0$ in V', W'. The polar of V' with regard to $\Sigma' = 0$ passes through U' and meets $V'W'$ in a point W''. By what we have just proved, a conic passes through U, V, W, U', V', W''. This conic has five points in common with $S = 0$ and therefore is $S = 0$, hence W'' is W'. Thus $U'V'W'$ is self-polar with regard to $\Sigma' = 0$, and, since U' has been chosen arbitrarily on $S = 0$, the theorem is proved.

Ex. 1. Referring to the text, prove that BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ touch a conic.

Ex. 2. If ABC, $A'B'C'$ are two triangles inscribed in a conic, prove that there exists a conic with regard to which both triangles are self-polar. And consider the dual of this theorem.

(iii) Pencils of outpolar conics.—Let $S_1 = 0$, $S_2 = 0$ be two conics each outpolar to the conic-envelope $\Sigma' = 0$. We prove that every conic of the pencil $S_1 + \lambda S_2 = 0$ is outpolar to $\Sigma' = 0$.

The proof is trivial by algebra since $\Theta_1' = 0$, $\Theta_2' = 0$ together imply $\Theta_1' + \lambda\Theta_2' = 0$.

Geometrically, let A be a base point of the pencil and let A be its polar relative to $\Sigma' = 0$. Then, by part (ii), $S_1 = 0$, $S_2 = 0$ meet A in pairs of points B_1, C_1 and B_2, C_2, each pair being conjugate with regard to $\Sigma' = 0$. Therefore the involution of pairs of points in which A meets the conics of the pencil is the same as the involution of pairs of points on A which are conjugate with regard to $\Sigma' = 0$. Hence, if any conic of the pencil meets A in B, C, the triangle ABC is self-polar relative to $\Sigma' = 0$; and therefore the conic is outpolar to $\Sigma' = 0$

If the base points of the pencil are distinct, they form a quadrangle $AA'A''A'''$ whose pairs of opposite sides are pairs of conjugate lines with regard to $\Sigma' = 0$; and it will be recognised that Hesse's theorem is a particular case of the theorem just proved. Such a quadrangle is called a *polar quadrangle* relative to $\Sigma' = 0$. A restatement of the theorem proved is that every conic passing through the vertices of a polar quadrangle relative to $\Sigma' = 0$ is outpolar to $\Sigma' = 0$

Ex. 3. If $ABCD$ and $ABC'D'$ are two polar quadrangles relative to $\Sigma' = 0$, then A, B, C, D, C', D' lie on a conic.

Ex. 4. If ABC is a triangle self-polar with regard to $\Sigma' = 0$ and D is any other point, then $ABCD$ is a polar quadrangle.

(iv) Gaskin's theorem.—As an application of the theorem in part (iii) we prove Gaskin's theorem which asserts that the circumcircle of a triangle, which is self-polar with regard to a given conic, is orthogonal to the director-circle of the conic.

Consider two such circles, intersecting in A, B and in the circular points I, J. These circles determine a co-axal system whose limiting points are $C \equiv AI\,.\,BJ$ and $D \equiv AJ\,.\,BI$. CI, CJ are then conjugate with regard to the conic; therefore the director circle passes through C, similarly, it passes through D. The director-circle therefore belongs to the conjugate co-axal system and is therefore orthogonal to the given circles.

46. Remarks on Chapter V.

There is little to add to what has been said already in the text. The problems considered consist of a set of very interesting applications of the principles developed in the first three chapters of this book.

The orthogonal systems of curves which have been obtained should be considered in relation to the proposition that, if x, y, u, v are real and $x + iy = \phi(u + iv)$, where ϕ is a holomorphic function, then the curves, in the real euclidean plane in which x, y are rectangular distance co-ordinates, given by $u =$ constant

intersect orthogonally the curves given by $v =$ constant. In particular the following cases should be examined.

Ex. 1. If $x + iy = u + iv$, the curves $u =$ constant, $v =$ constant are orthogonal systems of lines.

Ex. 2. If $(x + iy)^2 = u + iv$, the curves $u =$ constant, $v =$ constant are two orthogonal systems of rectangular hyperbolas.

Ex. 3 If $x + iy = a \cos (u + iv)$, the curves $u =$ constant are the hyperbolas and the curves $v =$ constant are the ellipses of a confocal system.

Ex. 4. If $x + iy = a \tan (u + iv)$, the curves $u =$ constant, $v =$ constant are conjugate systems of co-axal circles.

Such orthogonal systems of curves, it may be mentioned, find place in the theory of electrostatics, steady flow of electrical currents, magnetic fields, and hydrodynamics.

The theory of reciprocation is referred to in the next chapter in connection with the more general theory of correlations. Perhaps it may be said that interest in reciprocation is mainly concerned with the metrical considerations arising when we reciprocate with regard to a circle, and in these applications of the theory geometrical methods prevail over algebraic methods.

In connection with the theory of outpolar conics it is to be noted that a general linear condition imposed on the coefficients in the equation of a conic is equivalent to making the conic outpolar to a certain conic-envelope, and a corresponding remark applies to inpolarity

In the usual notation, a single linear condition

$$aA' + bB' + cC' + 2fF' + 2gG' + 2hH' = 0$$

on the coefficients of a conic $S = 0$ is equivalent to the condition that the conic should pass through a certain point $P\ (x_1, y_1, z_1)$ if and only if

$$A' : B' : C' : F' : G' : H' = x_1^2 : y_1^2 : z_1^2 . y_1z_1 : z_1x_1 : x_1y_1,$$

that is if and only if the equation of the conic-envelope $\Sigma' = 0$ may be written in the form

$$(lx_1 + my_1 + nz_1)^2 = 0.$$

Thus the condition that the conic should pass through P may be expressed by saying that the conic should be outpolar to the conic-envelope consisting of the pencil of lines with vertex at P counted twice.

The conics outpolar to four general conic-envelopes $\Sigma_1' = 0$, $\Sigma_2' = 0$, $\Sigma_3' = 0$, $\Sigma_4' = 0$ form a pencil; and every conic of the pencil is outpolar to every conic-envelope of the system $\lambda_1\Sigma_1' + \lambda_2\Sigma_2' + \lambda_3\Sigma_3' + \lambda_4\Sigma_4' = 0$. The conics of the pencil

have four common points, which equally determine the pencil; therefore the linear conditions imposed by these four points are equivalent in aggregate to the four outpolarity conditions defining the pencil and, moreover, must each be obtainable as a linear combination of the outpolarity conditions. Thus the system $\lambda_1\Sigma_1' + \lambda_2\Sigma_2' + \lambda_3\Sigma_3' + \lambda_4\Sigma_4' = 0$ contains just four members each consisting of a pencil of lines counted twice; the vertices of these pencils of lines are the base points of the pencil of conics. We remark further that there are just three pairs of lines such that each pair is conjugate with regard to all the members of the system $\lambda_1\Sigma_1' + \lambda_2\Sigma_2' + \lambda_3\Sigma_3' + \lambda_4\Sigma_4' = 0$; these are the line-pairs in the pencil of conics.

CHAPTER VI

COLLINEATIONS

47. Collineations.

(i) **Definition.**—We now consider in detail the simplest aspect of the theory of correspondences between the points of two modified complex euclidean planes π, π'; these planes may, in particular, coincide.

Let there be a (1, 1) correspondence between the points of π and π' in which the points of every line in π correspond to the points of a line in π' and *vice versa*. The correspondence determines two transformations; one, T, transforms any point P of π into the corresponding point $P' = T(P)$ in π'; the other, T^{-1}, called the inverse of T, transforms P' into $P = T^{-1}(P')$. T, as also T^{-1}, is called a *collineation*.

The points on corresponding lines are in (1, 1) correspondence; if this subsidiary correspondence, which we describe as being *induced* by T, is projective for all pairs of corresponding lines, the collineation T is called a *homography* or *projective collineation*; T^{-1} is then also projective.

A collineation evidently sets up a (1, 1) correspondence between the lines of π and those of π'; every pencil of lines in π corresponds to a pencil in π' and *vice versa*. The correspondence between corresponding pencils is projective if and only if the collineation is projective.

Harmonic relations between elements in π clearly correspond, under any collineation, to the same harmonic relations between the corresponding elements in π'.

The points of an irreducible conic in π correspond, under any collineation, to the points of an irreducible conic in π'; and a projective collineation induces a projectivity between the points of the two conics. A similar statement applies to irreducible conic-envelopes.

A collineation between two (possibly coincident) modified real euclidean planes is defined exactly in the same way as in the complex case. Clearly a collineation between two complex planes induces a collineation between the embedded real planes. In section 49 (iii) we prove a theorem, due to von Staudt, that every collineation between real planes is necessarily projective. Since we are able to give, below, an example of a non-projective collineation between complex planes, we observe that there is an important distinction between the real and complex cases.

(ii) Examples of projective and non-projective collineations.

(*a*) We may assume that the reader will generalise the idea of a modified complex euclidean plane to three dimensions and arrive at the notion of a modified complex euclidean space.

Then the simplest example of a projective collineation between two different planes, both contained in such a space of three dimensions, is obtained by defining corresponding points P, P' to be points in line with a given point O which does not belong to either plane (Fig. 88).

(*b*) If the planes π, π' coincide, the definition of projective collineation is satisfied by making every point correspond to itself; we have the projective collineation *identity*.

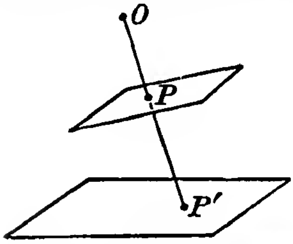


FIG 88 — PERSPECTIVE BETWEEN TWO PLANES

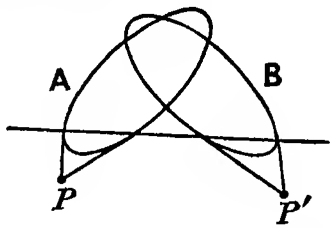


FIG 89 — POINTS CORRESPONDING IN A PARTICULAR COLLINEATION

For a less trivial example of a projective collineation between coincident planes, we may take a different plane π'', contained with π in a space of three dimensions, and set up, as in (*a*), a collineation between the points P of π and P'' of π'' and another collineation between the points P'' of π'' and P' of π'. The correspondence between P and P' clearly determines a projective collineation.

But we need not go outside the plane $\pi = \pi'$ for an example Consider two irreducible conics A, B in the plane and make P correspond to P' when the polar of P relative to A coincides with the polar of P' relative to B. P is transformed into P' by a projective collineation.

(*c*) With arbitrarily chosen systems of reference in the (distinct or coincident) complex planes π, π', with respect to which coordinates are x, y, z and x', y', z', a collineation is determined by the equations

$$x' : y' : z' = \bar{x} : \bar{y} : \bar{z},$$

where x, $\bar{x}$ are conjugate complex numbers, and so on. This collineation is not projective, as we may see by observing that corresponding points on the corresponding lines $x = 0$, $x' = 0$

have co-ordinates of the form $(0, 1, \theta)$, $(0, 1, \bar{\theta})$; the parameters θ, $\bar{\theta}$ do not satisfy any non-singular bilinear equation. Or we may observe that a cross-ratio is transformed by the collineation into the conjugate complex cross-ratio

(iii) Projective collineations.—In each of the planes π, π' (where π' may be π) let us take a system of reference (the two systems not necessarily being the same when π' is π). It is convenient to use co-ordinates x_1, x_2, x_3, instead of the more customary x, y, z, in π, and x_1', x_2', x_3' in π'.

Let us consider the equations

$$\begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned}$$

where $\rho \neq 0$ and

$$0 \neq \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

For shortness we write these equations in the form

$$\rho x_i' = a_{ij}x_j,$$

where it is intended, according to a commonly accepted *summation convention*, that terms involving a repeated suffix are summed with respect to that suffix (thus $a_{ij}x_j$ stands for $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$); and we denote the matrix of the coefficients a_{ij} by (a) and their determinant by $|a|$.

Since $|a| \neq 0$, the equations are reversible; we have

$$\rho' x_i = A_{ji}x_j',$$

where $\rho' \neq 0$ and A_{ji} is the cofactor of a_{ji} in $|a|$.

The equations thus set up a (1, 1) correspondence between the points of π and π'. Moreover, the line $l_i x_i = 0$ in π corresponds to the line $l_i' x_i' = 0$ in π', where

$$\sigma l_i' = A_{ij} l_j \qquad (\sigma \neq 0),$$

or, equivalently,

$$\sigma' l_i = a_{ji} l_j' \qquad (\sigma' \neq 0).$$

Thus points correspond to points and lines to lines, both in (1, 1) manner.

The point $(u_i + \lambda v_i)$, this symbol standing for a set of three co-ordinates, corresponds to the point $(u_i' + \lambda v_i')$, where $\rho u_i' = a_{ij}u_j$ and $\rho v_i' = a_{ij}v_j$. As λ varies, the two points describe corresponding lines and the two points correspond projectively on these lines. This remark applies to every pair of corresponding lines.

The correspondence represented by the linear equations therefore determines a projective collineation. For the remainder of this section it is convenient to know that the converse of this statement is true; the proof is given in section 49 (ii).

Because of this equivalence between projective collineations and sets of three linear equations we intend (except in regard to von Staudt's theorem in section 49 (iii)) to restrict our attention to those collineations which are projective. Accordingly, we may now agree, for brevity, that "collineation" is always to be taken to mean "projective collineation" in what follows.

(iv) The group of collineations in one plane.—If S, T are two collineations, operating in one plane π, such that $T(P) = P'$, $S(P') = P''$, it is clear that P corresponds to P'' in a new collineation; this collineation is denoted by ST and is called the *product* of T by S, $P'' = ST(P)$. Since $S^{-1}(P'') = P'$, $T^{-1}(P') = P$, the inverse of ST is $T^{-1}S^{-1}$.

Granting that every (projective) collineation may be represented by a triad of linear equations as in part (iii) above, or, what comes to the same thing, by an equality between ratios

$$x_1' : x_2' : x_3' = a_{1j}x_j : a_{2j}x_j : a_{3j}x_j,$$

we may prove, exactly as in section 9 (iii), that the collineations in π form a group with respect to the above definition of multiplication; and that the inverse of a collineation as defined above is the same as the inverse in the group sense. As usual the collineation identity is denoted by I.

The group of collineations is not abelian. This may be inferred from the following exercises.

Ex. 1. If S is given by $x' : y' : z' = ax : y : z$ and T is given by $x' : y' : z' = x : by : z$, show that ST and TS are both given by $x' : y' : z' = ax : by : z$

Ex. 2. If S is given by $x' : y' : z' = ax : y : z$ and T by $x' : y' : z' = x + bz : y : z$, show that ST is given by $x' : y' : z' = ax + abz : y : z$ and that TS is given by $x' : y' : z' = ax + bz : y : z$.

(v) Invariant elements of a collineation in one plane.

(*a*) A point which corresponds to itself in a collineation in one plane may be called a *united point* of the collineation, according to the terminology of projective transformations on a line; but we describe it now as *invariant under the collineation*. A point not corresponding to itself is *variant*.

A line may correspond to itself in one or other of two ways. First, every point on the line may be invariant; we then say that the line is *totally invariant*. Otherwise, it may correspond

to itself in such a way that its points correspond in a projectivity which is not identity; we then say that the line is *simply invariant*; it contains two invariant points, these coinciding if the projectivity on the line is parabolic. A collineation for which the inaccessible line is invariant is called *affine*.

Similarly, a pencil of lines may be invariant; if every line of it is invariant, the pencil is *totally invariant*; otherwise the pencil is *simply invariant*, in which case it has two (possibly coincident) invariant lines. Every point on every line of a pencil is invariant if and only if the collineation is identity.

(*b*) A conic may be invariant under a collineation. (For example, the conic $xz - y^2 = 0$ is invariant under the collineation given by $x' : y' : z' = x : -y : z$, both sets of co-ordinates referring to the same frame.)

Let the variable point P correspond to P' on the conic; and let A be any given point on the conic. The point A', corresponding to A, is also on the conic. It is a property of the conic that $A(P') \overline{\wedge} A'(P')$; and a property of the collineation that $A'(P') \overline{\wedge} A(P)$. Hence $A(P) \overline{\wedge} A(P')$. From this it follows that, in any parameterisation, the parameters of P, P' are connected by a non-singular bilinear equation. Hence, either every point of the conic is invariant or else just two (possibly coincident) points on it are invariant.

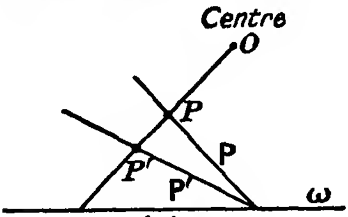


FIG 90—HOMOLOGY.

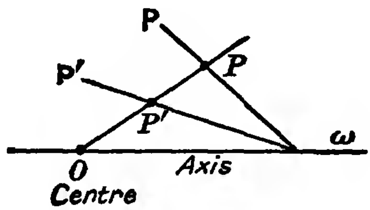


FIG 91—ELATION.

48. Perspective collineations.

(1) Definitions and elementary properties.—A collineation in one plane which leaves invariant every point of a given line ω and every line through a given point O is called a *perspective collineation* or *plane perspective*. O is called the *centre* and ω the *axis* of the perspective. If O is not on ω, the perspective is called a *homology* (Fig. 90); if O is on ω, it is called an *elation* (Fig. 91).

It is evident that the collineation identity is both a homology and an elation. We prove that a necessary and sufficient condition for a perspective to be identity is that at least one point A, not O nor on ω, should be invariant.

Let P be any point different from those already specified. The line AP has invariant points at A and $AP\,.\,\omega$ and is therefore invariant. If P is not on AO, then P is the intersection of the invariant lines AP, OP and is therefore invariant. If P is not on AO, let Q be any point not already specified and not on AO; then by what we have just observed, Q is invariant and so is QP, therefore, again, P is invariant. O is obviously invariant Therefore every point in the plane is invariant, that is the perspective is identity.

The condition is therefore sufficient, its necessity is clear.

Therefore, in the case of a homology, other than identity, every line through O is simply invariant, and the projectivity induced on each such line has distinct united points, at O and where the line meets ω. And in the case of an elation, every line through O, except ω, is simply invariant, and the projectivity induced on each such line is parabolic, having its coincident united points at O.

In both cases, corresponding points P, P' are in line with O, and corresponding lines P, P' meet on ω.

(ii) A perspective is determined by its centre, axis, and one pair of corresponding points.—We now prove that a perspective is determined uniquely when the centre O, axis ω, and one pair of non-coincident corresponding points A, A' are given.

Let, then, P be any point in the plane not already specified we have to show that its corresponding point P' is determined uniquely. If P is not on OA, P' is the point on OP such that AP, $A'P'$ meet on ω, and, if P is on OA, P' is the point on OP such that QP, $Q'P'$ meet on ω, Q, Q' being any pair of distinct corresponding points not on OA (by Desargues' theorem, P' is independent of Q, Q').

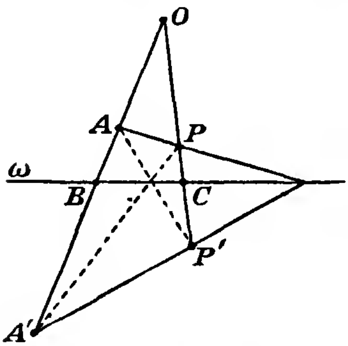


FIG 92 —HARMONIC HOMOLOGY.

It should be remarked, in the case of a homology, that if OAA' meets ω in B and OPP' meets ω in C, then, by what has just been said, if O, B *harm* A, A', then O, C *harm* P, P' (Fig. 92). Such a homology T is called a *harmonic homology*; it is uniquely determined by O, ω; and it has period 2; that is $T^2(P) \equiv P$ for all positions of P.

(iii) Equations of a homology.—We prove that a homology may be represented by equations of the form

$$x' : y' : z' = ax : y : z.$$

We choose a system of reference so that O is assigned the co-ordinates (1, 0, 0) and ω the equation $x = 0$. Let A, A' be two non-coincident corresponding points; to these we may assign co-ordinates of the forms (x_0, y_0, z_0), (ax_0, y_0, z_0) with $a \neq 0, \infty$ and $x_0 \neq 0$. Let corresponding points P, P' have co-ordinates (x, y, z) and (x', y', z') respectively.

First, suppose that P is not on OA, and let AP meet ω at R. Then R is represented by either of the symbols

$$xA - x_0P \equiv (xy_0 - x_0y)B + (xz_0 - x_0z)C,$$

where OBC is the triangle of reference and B, C are assigned the co-ordinates (0, 1, 0), (0, 0, 1) respectively. Also, since $A'P'$ passes through R, R is also represented by the symbol

$$x'A' - ax_0P' \equiv (x'y_0 - ax_0y')B + (x'z_0 - ax_0z')C.$$

Therefore

$$\frac{xy_0 - x_0y}{x'y_0 - ax_0y'} = \frac{xz_0 - x_0z}{x'z_0 - ax_0z'}.$$

Since also $$y' : z' = y : z,$$

we have $$x' : y' : z' = ax : y : z.$$

If P is on OA, the same result is reached by using, instead of A, A', two distinct corresponding points Q, Q', not on OA; by what we have just proved, these have co-ordinates of the form (x_1, y_1, z_1), (ax_1, y_1, z_1) respectively.

Finally, we observe that the invariant points at O and on ω have co-ordinates which satisfy $x' : y' . z' = ax : y : z$. This completes the proof.

The equations show at once that the collineation identity arises when $a = 1$, the equations then being

$$x' : y' : z' = x : y : z.$$

And the homology has period 2 if and only if $a^2 = 1$, $a \neq 1$, that is if and only if $a = -1$; this is the case of the harmonic homology, and the appropriate equations are

$$x' : y' : z' = -x . y : z.$$

(iv) Equations of an elation.—Next we prove that an elation may be represented by equations of the form

$$x' : y' : z' = x + az : y : z.$$

This may be done as in part (iii), but it will be more useful to indicate an alternative method which also applies in the case of a homology.

We take O as (1, 0, 0) and ω as $z = 0$. Then, if $P\,(x, y, z)$

corresponds to P' (x', y', z'), we must have $y' : z' = y : z$ since P, P' are in line with O, and, when $z = 0, z' = 0$ and $x' : y' = x : y$, since ω is totally invariant.

These conditions are satisfied by the equations written above. Moreover, these equations represent a collineation; and, in view of the conditions mentioned, the collineation is an elation with centre O and axis ω

To show that every elation with the assigned centre and axis has equations of this form, it is enough, by the uniqueness theorem of part (ii), to show that we can find a so that any assigned pair of distinct points A, A', in line with but different from O, correspond. In fact, A, A' must have co-ordinates of the form (x_0, y_0, z_0), (kx_0, y_0, z_0) respectively, if $x_0 \neq 0$, with $k \neq \infty$, $z_0 \neq 0$; we have therefore only to take

$$a = (k - 1)x_0 z_0^{-1}.$$

And if $x_0 = 0$, A, A' have co-ordinates of the form $(0, y_0, z_0)$, (x_0, y_0, z_0), with $x_0 z_0 \neq 0$, and we take

$$a = x_0 z_0^{-1}.$$

There is no periodic elation except identity (given by $a = 0$), this having period 1

Ex. 1. Prove the theorem of part (iii) of this section by the method of part (iv) and the theorem of part (iv) by the method of part (iii).

(v) Metrical forms of perspectives.

(*a*) A homology is said to be *homothetic* when the centre O is accessible and the axis ω is inaccessible. Homothetic figures are similar since the lines which determine any angle in one figure are parallel to the lines which determine the corresponding angle in the other figure. In regard to the embedded real plane, if O is real and if the real accessible points P, A correspond to real accessible points P', A', then $\overrightarrow{OP'}/\overrightarrow{OP} = \overrightarrow{OA'}/\overrightarrow{OA}$; that is $\overrightarrow{OP'}/\overrightarrow{OP}$ is constant (Fig. 93). By reason of this property, a homothetic homology is also called a *dilatation*.

A harmonic homothetic harmology has the property that O is the mid point of any pair of corresponding points P, P' (Fig. 94); the homology is called *reflection in* O. When O is real, corresponding figures in the embedded real plane are congruent.

(*b*) Consider a homology whose axis ω is accessible and whose centre O is inaccessible. In regard to the embedded real plane, if ω and O are real and the line joining corresponding real points

P, P' meets ω at R, then $\overrightarrow{PR}/\overrightarrow{RP'}$ is constant. By reason of this property, such a homology is also called a *uniform stretch* in the direction associated with O

In the case of a harmonic uniform stretch, the mid point of any pair of corresponding points P, P' is on the axis. And if, further, O and the inaccessible point on ω are harmonic with regard to the circular points, PP' is perpendicular to ω; the homology is called *reflection in ω* (Fig. 95).

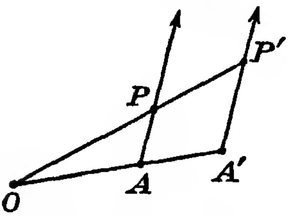


FIG 93.—DILATATION.

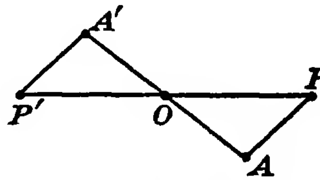


FIG. 94.—REFLECTION IN THE CENTRE.

(c) If an elation has its axis (and therefore its centre) inaccessible, then any two pairs of corresponding points P, P' and Q, Q' form the vertices of a parallelogram with the two sides PP' and QQ' in a fixed direction. In regard to the embedded real plane

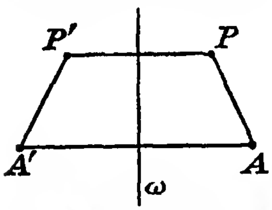


FIG. 95.—REFLECTION IN THE AXIS

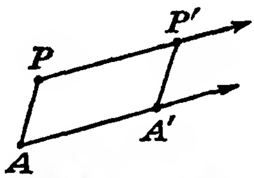


FIG. 96 —TRANSLATION.

(Fig. 96) if O is real and if P, P' are corresponding accessible real points, $\overrightarrow{PP'}$ is constant and corresponding figures are congruent. By reason of this property, an elation with inaccessible axis is called a *translation* in the direction corresponding to the centre.

49. Fundamental theorems on collineations.

(i) A unique collineation transforms a given quadrangle into another given quadrangle.—We prove that there is a unique collineation which transforms any given quadrangle $ABCD$ into any given quadrangle $A'B'C'D'$ of the same or of a different plane. In doing so, we show that the collineation may be expressed as the product of a finite number of perspectives.

This *theorem of uniqueness* is most useful; it should be considered in relation to the theorem of section 48 (ii).

If the two quadrangles are in the same plane and A is not A' let O_1 be any point on AA', different from A, A', and let ω_1 be any line not through A or A' (Fig. 97) Then there is a perspective T_1 with O_1 as centre and ω_1 as axis which transforms A into A'. Let T_1 transform B, C, D into B_1, C_1, D_1 respectively.

Let O_2 be any point on B_1B', different from B_1, B', and let ω_2 be any line through A' but not through B_1 or B'. There is a perspective T_2 with O_2 as centre and ω_2 as axis which transforms A', B_1 into A', B'. Let T_2 transform C_1, D_1 into C_2, D_2.

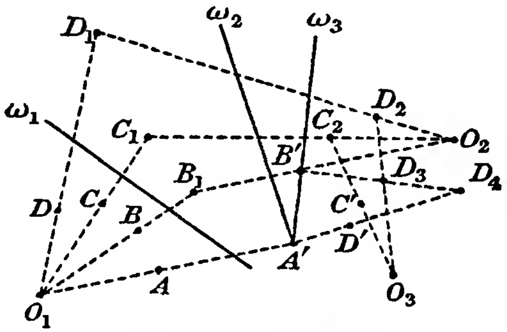


FIG 97 —FIGURE FOR THE THEOREM IN SECTION 49 (i).

Let O_3 be any point on C_2C', but not C_2 or C', and let T_3 be the perspective with O_3 as centre and $A'B'$ as axis which transforms C_2 into C' It is necessary to remark that C' is not on $A'B'$, nor is C_2, for otherwise C_1 would be on $A'B_1$ and therefore C would be on AB. Let T_3 transform D_2 into D_3. D_3 cannot be on a side of the triangle $A'B'C'$, for otherwise D_2 would be on a side of $A'B_2C_2$, D_1 would be on a side of $A'B_1C_1$, and D would be on a side of ABC.

Let $A'D'$, $B'D_3$ meet at D_4. Then there is a perspective T_4, with centre B' and axis $A'C'$, which transforms D_3 into D_4; and there is a perspective T_5, with centre A' and axis $B'C'$, which transforms D_4 into D'.

The collineation $T_5T_4T_3T_2T_1$ transforms A, B, C, D into A', B', C', D' respectively. It is a product of five plane perspectives, three of which may each be selected in an infinity of ways

If the second quadrangle lies in a different plane from the first, both planes lying in a space of three dimensions, only one extra step is required. Let O be any point outside both planes; the correspondence between a point P of the first plane and a point P' of the second plane, when O, P, P' are in line, is called a *space*

perspective; clearly it is a collineation between the two planes. Then we have only to project $ABCD$ from O onto the second plane and apply the construction just described to this new quadrangle.

Having shown the existence of a collineation which has the required properties, we must next prove that it is unique. For this it is enough to show that, given A, B, C, D, P and A', B', C', D', P', the point P', which corresponds to P in any collineation transforming A, B, C, D into A', B', C', D', is determined uniquely.

Now, in fact, if P is not on any side of the quadrangle $ABCD$, P' is uniquely determined as the intersection of $A'P'$, $B'P'$ where $A(B, C, D, P) \barwedge A'(B', C', D', P')$ and $B(A, C, D, P) \barwedge B'(A', C', D', P')$. If P is on the side CD, opposite to AB, the above reasoning is still valid if P is not at C, D or $AB \,.\, CD$. The argument serves, *mutatis mutandis*, when P is on any other side of the quadrangle but not at a vertex. If P is at A, B, C or D, P' is at A', B', C' or D' respectively If P is at $AB \,.\, CD$, P' is at $A'B' \,.\, C'D'$, and so on.

Ex. 1. Prove that a collineation which leaves the four points of a quadrangle each invariant is identity; and deduce that a collineation which makes two quadrangles correspond is unique

(ii) Equations of a collineation.—Equations for the collineation which transforms A, B, C, D into A', B', C', D' are now easily obtained. If we take two systems of reference, one with ABC as triangle of reference and D as unit point, and the other with $A'B'C'$ as triangle of reference and D' as unit point, and rely on the uniqueness theorem, the equations are at once seen to be

$$\rho x_i' = x_i, \quad i = 1, \ 2, 3, \rho \neq 0.$$

If the two planes are the same, the collineation may still be represented by a set of linear equations in which both sets of co-ordinates are attached to a single system of reference. Referring to the systems just mentioned, let (x_i'') be the co-ordinates, relative to the first system, of the point which has co-ordinates (x_i') relative to the second system; these co-ordinates are connected by equations of the form

$$\rho' x_i'' = a_{ij} x_j', \quad (\rho' \neq 0, |a| \neq 0).$$

Then, relative to the first system, the equations of the collineation are

$$\rho'' x_i'' = a_{ij} x_j, \quad \rho'' = \rho' \rho.$$

Ex. 2. Find the equations of the collineation which transforms the quadrangle (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) into the quadrangle (a_i), (b_i), (c_i), (d_i).

Ex. 3. A collineation of period 2 is a harmonic homology.

[Let P be a variant point and P' its corresponding point. Then there must be another point Q, not on PP', which is also variant (otherwise we could easily show that P would be invariant); let Q' be its corresponding point. Q' is not on PP' since Q is not thereon and the collineation is of period 2. Let PP' meet QQ' at O; and let O, X *harm* P, P' and O, Y *harm* Q, Q'. Then the harmonic homology with centre O and axis XY transforms $PP'QQ'$ into $P'PQ'Q$; by the uniqueness theorem it is the given collineation.]

Taking P, P', Q, Q' to have co-ordinates (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) respectively, show that the equations of the collineation are

$$x' : y' : z' = -y + z : -x + z : z,$$

and verify algebraically that the period is 2.

Ex. 4. Find the equations of the collineation which transforms A (1, 0, 0), B (0, 1, 0), C (0, 0, 1), D (1, 1, 1) into B, C, D, A respectively. This collineation has period 4; verify this algebraically.

(iii) Von Staudt's theorem for a real plane.—The reader is reminded that, for the sake of abbreviation, we decided in section 47 (iii) to use the word "collineation" to mean "projective collineation." This abbreviation has necessarily to be abandoned for the following discussion but is resumed in the next section.

We prove the theorem of von Staudt that every collineation (now understanding the word in its wide sense) in a modified real euclidean plane is a projective collineation.

Let K be any given collineation in the plane and let $ABCD$ be any given quadrangle. Then the points $K(A)$, $K(B)$, $K(C)$, $K(D)$ are also the vertices of a quadrangle.

By part (i) of this section, there is a projective collineation T which transforms $K(A)$, $K(B)$, $K(C)$, $K(D)$ into A, B, C, D respectively. The collineation TK, this product being defined as in section 47 (iv), therefore has four invariant points, at A, B, C, D. We prove that $TK = I$, the collineation identity. Since collineations (in the wide sense) obviously form a group with respect to the same rule of combination as for projective collineations, we can then infer that $K = T^{-1}$, that is that K is a projective collineation.

To prove that $TK = I$, it is enough to prove that every point on each of AB, BC is invariant; for then, if P is any point not on these lines and if we take through P any two lines not containing B, both lines are invariant since their intersections with AB, AC are invariant; consequently their common point P is invariant. This proof is carried out only for the line BC since the argument applies, *mutatis mutandis*, to AB.

On BC we have already three points invariant under TK, namely B, C and the intersection of BC with AD. Let us take a (real) co-ordinate system with ABC as triangle of reference and D as unit point; and let TK transform the point $(0, 1, \theta)$ into the point $(0, 1, f(\theta))$, where $f(\theta)$ is a real function of θ. Then we have

$$f(0) = 0, f(1) = 1, f(\infty) = \infty.$$

Since harmonic relations are invariant under TK, it follows that $\{f(\theta_1), f(\theta_2); f(\theta_3), f(\theta_4)\} = -1$ if and only if $\{\theta_1, \theta_2; \theta_3, \theta_4\} = -1$. Hence, since $\{0, 2\theta; \theta, \infty\} = -1$, we have $\{0, f(2\theta); f(\theta), \infty\} = -1$; that is

$$f(2\theta) = 2f(\theta).$$

And, since $\{\theta, \phi; \tfrac{1}{2}(\theta + \phi), \infty\} = -1$, we have

$$\{f(\theta), f(\phi); f(\tfrac{1}{2}(\theta + \phi)), \infty\} = -1;$$

that is
$$f(\theta) + f(\phi) = 2f(\tfrac{1}{2}(\theta + \phi)).$$

Combining these results,

$$f(\theta) + f(\phi) = f(\theta + \phi).$$

Hence, for any integer $n > 0$,

$$f(\theta) + f((n-1)\theta) = f(n\theta)$$

and so, inductively,

$$nf(\theta) = f(n\theta);$$

and then
$$f(-n\theta) + f(n\theta) = f(0) = 0,$$

so that
$$f(-n\theta) = -f(n\theta) = -nf(\theta).$$

Hence, for any non-zero positive or negative integers p, q,

$$qf(p/q) = f(p) = pf(1) = p;$$

thus, for all rational values of θ,

$$f(\theta) = \theta.$$

Our object is to show that this last result holds also for all irrational values of θ. For this purpose we show that $f(\theta)$ is a non-decreasing function of θ. Since $\{\theta, -\theta; 1, \theta^2\} = -1$, we have $\{f(\theta), f(-\theta); 1, f(\theta^2)\} = -1$; that is

$$f(\theta^2) = [f(\theta)]^2.$$

Hence, if $\lambda (= \mu^2)$ is any positive number,

$$f(\theta + \lambda) = f(\theta) + f(\lambda) = f(\theta) + [f(\mu)]^2;$$

that is to say $f(\theta)$ is non-decreasing.

If, now, θ is irrational, for every rational number $\theta_1 < \theta$ and every rational number $\theta_2 > \theta$ we have

$$\theta_1 = f(\theta_1) \leqslant f(\theta) \leqslant f(\theta_2) = \theta_2,$$

that is

$$\theta_1 \leqslant f(\theta) \leqslant \theta_2;$$

therefore $f(\theta) = \theta$.

The proof of our theorem is now completed.

Ex 5 Prove that any collineation between two different modified real euclidean planes is projective.

50. Invariant points of a collineation in one plane.

(i) A collineation having three invariant points in line is a perspective.—We restrict ourselves again to projective collineations

We have already remarked that a collineation which has four invariant points, of which no three are in line, is identity.

We prove now that a collineation, not identity, which has three invariant points in line, is a perspective.

The collineation induces on the line a projectivity with three united points which is therefore identity; every point on the line is therefore united.

Taking the line to be $x = 0$, the equations of the collineation have the form

$$x' : y' \cdot z' = a_1 x : a_2 x + b_2 y : a_3 x + b_2 z.$$

Let O be the point $(a_1 - b_2, a_2, a_3)$ Then, if P, P' are corresponding points, their join PP' meeting $x = 0$ at Q, we have bonds

$$O \equiv P' - b_2 P, \quad Q \equiv P' - a_1 P,$$

$$\{O, Q;\ P, P'\} = a_1/b_2.$$

Therefore P, P' correspond in a perspective with centre O and axis $x = 0$.

Ex. 1. Obtain this result geometrically.

(ii) A collineation, not identity nor a perspective, has three invariant points, not in line.—We next prove that a collineation, which is neither identity nor a perspective, has three invariant points, by part (i), these points are not in line. We describe such a collineation as *general.*

There is at least one variant point P; let it be transformed by the collineation into P'; and let P' be transformed into P''. All three points are different.

Corresponding lines through P, P' intersect in the points of a

conic K_1, and corresponding lines through P', P'' intersect in the points of a conic K_1', the transform of K_1 (Fig. 98).

There are two possibilities to consider, according to whether P, P', P'' are in line or not.

If P, P', P'' are on a line X, then the conic K_1 consists of X and another line Y; and K_1' consists of X and Y′, the transform of Y. If Y, Y′ are coincident, then Y is totally invariant, for if Q is a point on Y, $Q = PQ \,.\, P'Q$ corresponds to $Q = P'Q \,.\, P''Q$, the collineation would then be a perspective. Therefore Y, Y′ are different; let them meet at A. A is an invariant point and is the only one not on X. On X, the induced projectivity has two (possibly coincident) united points; these, with A, are the three invariant points of the collineation.

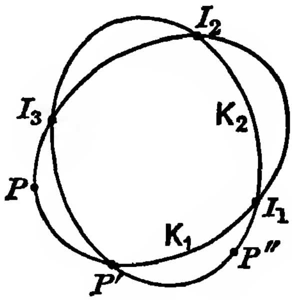


Fig 98.—Invariant Points of a Collineation

If P, P', P'' are not in line, the conics K_1, K_1' are both irreducible and meet at P' and in three other (possibly coincident) points A, B, C, each of which is invariant. There is no other invariant point, otherwise the conics would coincide, having five points in common, and the collineation would be identity.

(iii) Simplified forms for the equations of a collineation.—We are now in a position to put the equations of a collineation into simplified forms We have already done so in the case of a perspective, it remains to deal with the case of a collineation having three non-collinear invariant points I_1, I_2, I_3.

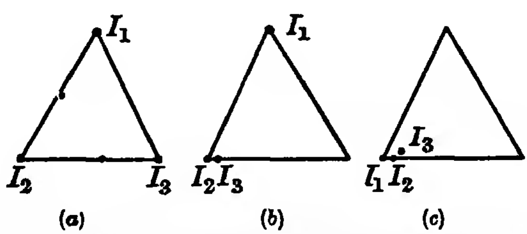


Fig. 99.—The Different Arrangements of the Invariant Points of a General Collineation

If the invariant points are distinct, we take them as the vertices of the triangle of reference (Fig. 99 (a)). Taking any other point, not on any side of this triangle, as unit point, let its

transform be (a, b, c). Then the equations of the collineation are

$$x' : y' : z' = ax \cdot by : cz.$$

In order that the collineation should not be a homology, a, b, c must be all different.

If I_2, I_3 are infinitely near (*cf.* section 25), that is, if the conics K_1, K_1' touch at I_2, the common tangent there being I_2I_3, we take I_1 as $(1, 0, 0)$, I_2 as $(0, 1, 0)$ and I_2I_3 as $x = 0$ (Fig. 99 (b)) Then $x = 0$ is transformed into $x' = 0$, $z = 0$ into $z' = 0$, and the point $(1, 0, 0)$ into itself. The equations therefore have the form

$$x' : y' : z' = a_{11}x : a_{22}y + a_{23}z : a_{33}z.$$

The projectivity induced on the line $x = 0$ is parabolic. The point $(0, 1, \lambda)$ is transformed into $(0, a_{22} + \lambda a_{23}, \lambda a_{33})$, that is, into $(0, 1, \mu)$, where $\mu(a_{22} + \lambda a_{23}) = a_{33}\lambda$. The united points of the projectivity are given by $\mu = \lambda$ and, by hypothesis, both arise when $\mu = \lambda = 0$. Hence $a_{22} = a_{33}$. The equations therefore take the form

$$x' : y' : z' = ax : by + cz : bz,$$

and we require $a \neq b$, since the collineation is not an elation.

If I_1, I_2, I_3 are infinitely near, that is, if the conics K_1, K_1' have three point contact, at I_1, we take I_1 as $(0, 1, 0)$ and I_1I_2 as $x = 0$ (Fig. 99 (c)). Since the conics are irreducible, I_3 does not lie on $x = 0$ Then $x = 0$ is transformed into $x' = 0$, and $(0, 1, 0)$ into itself. The equations therefore have the form

$$x' : y' : z' = a_{11}x : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{33}z.$$

The projectivity induced on $x = 0$ is parabolic, so, as before, $a_{22} = a_{33}$. Also the projectivity induced on the self-corresponding pencil with vertex I_1 is parabolic; therefore $a_{11} = a_{33}$. The equations therefore take the form

$$x' : y' : z' = ax : bx + ay + cz : dx + az.$$

(iv) Similitude.

(a) Among the collineations which are of metrical interest because they leave the inaccessible line at least simply invariant, we consider here those for which the pair of circular points I, J is invariant. Such collineations are divisible into two categories according as I, J are transformed into I, J respectively or into J, I respectively.

(b) First let us consider the collineations for which I, J are invariant points.

If the induced projectivity on IJ is identity, such a collineation is either a dilatation or a translation or is the collineation identity.

If the induced projectivity is not identity, such a collineation is either a homology with centre at I (or J) and axis through J (or I) or is a general collineation with two of its invariant points at I, J.

In any of these cases, let P be an accessible point and let U, V be two inaccessible points; then, if accents denote corresponding points,

$$P\{U, V; I, J\} = P'\{U', V', I, J\}.$$

Hence (section 10 (vi) (*b*)), corresponding figures are directly similar. The collineation is called a *direct similitude.*

With modified complex co-ordinates based on rectangular axes, let I, J have co-ordinates $(1, i, 0)$, $(1, -i, 0)$ respectively. Then the equations of a direct similitude have the form

$$x' : y' : z' = px + qy + rz : ux + vy + wz : z$$

with

$$\begin{aligned} 1 : i &= p + qi : u + vi, \\ 1 : -i &= p - qi : u - vi. \end{aligned}$$

Hence,

$$\begin{aligned} (q + u) - (p - v)i &= 0, \\ (q + u) + (p - v)i &= 0 \end{aligned}$$

and so

$$v = p, q = -u.$$

Therefore the equations of a direct similitude take the form

$$x' : y' : z' = px - uy + rz : ux + py + wz : z.$$

The equations represent a general collineation with three distinct invariant points, the third being at A $(\xi, \eta, 1)$, if and only if

$$p\xi - u\eta + r = \xi, u\xi + p\eta + w = \eta;$$

and then the equations may be written as

$$x' : y' : z'$$
$$= \xi z + p(x - \xi z) - u(y - \eta z) : \eta z + u(x - \xi z) + p(y - \eta z) : z.$$

Putting $p = R\cos\theta$, $u = R\sin\theta$ and using (X, Y), (X', Y') for the unmodified co-ordinates of corresponding accessible points, we have

$$\begin{aligned} X' - \xi &= R[(X - \xi)\cos\theta - (Y - \eta)\sin\theta], \\ Y' - \eta &= R[(X - \xi)\sin\theta + (Y - \eta)\cos\theta]. \end{aligned}$$

From this form of the equations, we see that the direct similitude is the product T_1T_2 of the commutative collineations T_1, given by

$$\begin{aligned} X_1 - \xi &= R(X - \xi), \\ Y_1 - \eta &= R(Y - \eta), \end{aligned}$$

and T_2, given by

$$\begin{aligned} X' - \xi &= (X_1 - \xi)\cos\theta - (Y_1 - \eta)\sin\theta, \\ Y' - \eta &= (X_1 - \xi)\sin\theta + (Y_1 - \eta)\cos\theta. \end{aligned}$$

T_1 is a dilatation with centre at A. If A is a real point and if θ is real, T_2 determines a positive rotation, about A and through the angle θ, of the embedded real plane; accordingly, it is convenient to describe T_2 as being, in all cases, a *positive rotation*, about A and through the interval θ, of the complex plane.

Ex. 2 Find the simplified forms of the equations

$$x' : y' . z' = px - uy + rz : ux + py + wz : z$$

which represent (i) a translation in the direction of the line $y = \lambda x$, (ii) a homology with centre I and axis $y + ix = 0$.

(*c*) Now let us consider the collineations which interchange I and J.

Such a collineation induces on IJ an involutory projectivity whose united points A, B are harmonic with regard to I, J. Therefore the collineation is either a harmonic homology with centre at A (or B) and axis through B (or A), that is a reflection, or is a general collineation with two invariant points at A, B.

With the notation of (*b*), we have, in either case,

$$P\{U, V, I, J\} = P'\{U', V'; J, I\}.$$

Hence, corresponding figures are contra-similar. The collineation is called a *contra-similitude.*

The equations of a contra-similitude may be shown, as in (*b*), to have the form

$$x' : y' : z' = px + uy + rz : ux - py + wz : z;$$

and when these equations represent a general collineation with its third invariant point accessible, at A $(\xi, \eta, 1)$, they may be put in the form

$$\begin{aligned} x' : y' : z' \\ = \xi z + p(x - \xi z) + u(y - \eta z) : \eta z + u(x - \xi z) - p(y - \eta z) : z \end{aligned}$$

Again putting $p = R\cos\theta$, $u = R\sin\theta$ and using (X, Y), (X', Y') for the unmodified co-ordinates of corresponding accessible points, we have

$$\begin{aligned} X' - \xi &= R[(X - \xi)\cos\theta + (Y - \eta)\sin\theta], \\ Y' - \eta &= R[(X - \xi)\sin\theta - (Y - \eta)\cos\theta] \end{aligned}$$

From this form of the equations, we see that the contra-similitude is the product $T_3T_2T_1$ of the collineations T_1, given by

$$\begin{aligned} X_1 - \xi &= \quad X - \xi, \\ Y_1 - \eta &= -(Y - \eta), \end{aligned}$$

T_2, given by

$$\begin{aligned} X_2 - \xi &= (X_1 - \xi)\cos\theta - (Y_1 - \eta)\sin\theta, \\ Y_2 - \eta &= (X_1 - \xi)\sin\theta + (Y_1 - \eta)\cos\theta, \end{aligned}$$

and T_3, given by

$$X' - \xi = R(X_2 - \xi),$$
$$Y' - \eta = R(Y_2 - \eta).$$

Thus, this contra-similitude is expressible as the product of a reflection by a direct similitude.

Ex. 3. Show that the contra-similitude considered in the text is also the product of the direct similitude T_2T_3 by the reflection in the line $Y - \eta = (X - \xi)\tan\theta$

(*d*) Let T_1, T_2 denote any two direct similitudes and S_1, S_2 denote any two contra-similitudes. By observing that the product of two collineations is a collineation and by considering the effects on the circular points of the various projectivities induced on the inaccessible line, we see that T_1T_2 and S_1S_2 are both direct similitudes and that T_1S_1 is a contra-similitude.

Hence, the direct similitudes and the contra-similitudes taken together form a group, called the *similarity group*, relative to the rule of multiplication for collineations; and the direct similitudes form a sub-group of the similarity group.

We use the term *similitude*, without qualification, to refer to a collineation which is specified to the extent of being either a direct similitude or a contra-similitude.

51. Miscellaneous properties of collineations.

Some interesting properties of collineations are contained in the following exercises.

Ex. 1. Two elations in one plane, which have a common axis, are commutative, and dually.

Ex. 2. Two homologies with the same centre and axis are commutative.

Ex. 3. A collineation with three distinct non-collinear invariant points is, in many ways, a product of three homologies.

Ex. 4. If T is an elation with centre O and if P is any point off the axis, O, P *harm* $T(P)$, $T^{-1}(P)$.

Ex. 5 If an irreducible conic is invariant under a homology, the homology is harmonic, and its centre and axis are respectively pole and polar relative to the conic.

Ex. 6. A collineation has three distinct invariant points I_1, I_2, I_3, not in line. P corresponds to P' and moves so that PP' passes through a fixed point K, not on any invariant line. Prove that P, P' generate two conics intersecting in I_1, I_2, I_3, K.

Ex. 7. Two irreducible conics K, K' are given in the same or different planes. On K take three distinct points A, B, C, the tangents at A, B intersecting at D; and on K' take three distinct

points A', B', C', the tangents at A', B' intersecting at D'. The collineation, which transforms A, B, C, D into A', B', C', D' respectively, *transforms* K *into* K$'$.

[Therefore we can find a collineation to transform a given irreducible conic and a given point off the conic into a circle and its centre. We have thus a means of converting a considerable number of well-known properties of a circle into theorems relating to an irreducible conic; and *vice versa*. The next exercise illustrates this notion.]

Ex. 8. The tangents from O to an irreducible conic K touch the conic at X, Y. P, Q are variable points on K such that $O\{X, Y; P, Q\}$ is constant. Prove that PQ touches a fixed conic which touches K at X, Y.

Ex. 9. Find the most general collineation which leaves invariant the conic $zx - y^2 = 0$.

[Parameterise the conic by the equations $x : y : z = \lambda^2 : \lambda : 1$. To every point, with parameter λ, corresponds a point, with parameter μ, where λ, μ are connected by a relation of the form

$$a\lambda\mu + b\lambda + c\mu + d = 0, \; ad \neq bc.$$

From this,

$$\mu^2 : \mu : 1 = (b\lambda + d)^2 \, . - (b\lambda + d)(a\lambda + c) : (a\lambda + c)^2.$$

This correspondence is clearly effected by the collineation

$$x' : y' : z' = b^2x + 2bdy + d^2z : - abx - (ad + bc)y - dcz \\ : a^2x + 2acy + c^2z.$$

By the uniqueness theorem, this is the only collineation which induces the same projectivity on the conic.

Verify that, for pairs of corresponding points on the conic,

$$a^2xx' + 2adyy' + d^2zz' = b^2xz' + 2bcyy' + c^2zx'.]$$

Ex. 10. Find the collineations which leave invariant the conic $X^2 + Y^2 = r^2Z^2$ and the point $(0, 0, 1)$.

[Put $X + iY = rx$, $X - iY = rz$, $Z = y$. Then we seek a collineation of the form in *Ex.* 9 which leaves invariant the point for which $x = 0$, $y = 1$, $z = 0$. We must therefore have

$$0 : 1 : 0 = 2bd : - (ad + bc) : 2ac, \; ad \neq bc.$$

The possibilities are either (i) $a = d = 0$, $b \neq 0$, $c \neq 0$, or (ii) $b = c = 0$, $a \neq 0$, $d \neq 0$.

In case (i) the collineation is

$$x' : y' : z' = b^2x : - bcy : c^2z$$

which is the same as

$$X' : Y' : Z' = \tfrac{1}{2}X(b^2 + c^2) + \tfrac{1}{2}iY(b^2 - c^2) \\ : - \tfrac{1}{2}iX(b^2 - c^2) + \tfrac{1}{2}Y(b^2 + c^2) : - bcZ.$$

Putting $b = -ce^{i\alpha}$, this takes the form

$$X' : Y' : Z' = X \cos\alpha - Y \sin\alpha : X \sin\alpha + Y \cos\alpha : Z.$$

In case (ii) the collineation is

$$x' : y' : z' = d^2z : -ady : a^2x,$$

which is the same as

$$X' : Y' : Z' = \tfrac{1}{2}X(a^2 + d^2) + \tfrac{1}{2}iY(a^2 - d^2)$$
$$: -\tfrac{1}{2}iX(d^2 - a^2) - \tfrac{1}{2}Y(d^2 + a^2) : -adZ.$$

Putting $d = -ae^{i\beta}$ this takes the form

$$X' : Y' : Z' = X \cos\beta + Y \sin\beta : X \sin\beta - Y \cos\beta : Z.$$

Metrically, what we have shown is that the collineations which leave a circle and its centre invariant are (i) a rotation about the centre, (ii) reflection in a diameter ($Y = X \tan \tfrac{1}{2}\beta$).

Reflection in the centre corresponds to a rotation with $\alpha = \pi$.]

52. The algebraic classification of collineations in one plane.

(i) The characteristic equation and matrix.—We take the equations of the collineation to be

$$x_1' : x_2' : x_3' = a_{1j}x_j : a_{2j}x_j . a_{3j}x_j,$$

with

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$$

the co-ordinates (x_i), (x_i') being attached to the same system of reference.

An invariant point is characterised by the possibility of finding τ ($\neq 0$) so that, at the point,

$$x_i' = \tau x_i, \quad i = 1, 2, 3.$$

The invariant points are therefore determined by the *fundamental equations*

$$\tau x_i = a_{ij}x_j, \; i = 1, 2, 3,$$

where τ is a root of the *characteristic equation*

$$\Delta(t) \equiv \begin{vmatrix} a_{11} - t & a_{12} & a_{13} \\ a_{21} & a_{22} - t & a_{23} \\ a_{31} & a_{32} & a_{33} - t \end{vmatrix} = 0.$$

The *characteristic matrix* of the equations of the collineation is

$$M(t) \equiv \begin{pmatrix} a_{11} - t & a_{12} & a_{13} \\ a_{21} & a_{22} - t & a_{23} \\ a_{31} & a_{32} & a_{33} - t \end{pmatrix}.$$

(ii) Intrinsic algebraic characters of a collineation.—We prove that, for any change of the co-ordinate system in the plane, the ratios of the roots of the characteristic equation and the rank of the characteristic matrix corresponding to each of these roots are unaltered. These ratios and ranks therefore represent intrinsic characters of the collineation itself.

On changing to a new system of reference, the points with the old co-ordinates (x_i), (x_i') acquire new co-ordinates $(\bar{x}_i)$, $(\bar{x}_i')$ respectively, given by equations of the form

$$\bar{x}_1 : \bar{x}_2 : \bar{x}_3 = b_{1r}x_r : b_{2r}x_r : b_{3r}x_r,$$
$$\bar{x}_1' : \bar{x}_2' : \bar{x}_3' = b_{1r}x_r' : b_{2r}x_r' : b_{3r}x_r',$$

with $|b| \neq 0$.

The new equations of the collineation are then

$$\bar{x}_1' : \bar{x}_2' : \bar{x}_3' = c_{1k}\bar{x}_k : c_{2k}\bar{x}_k : c_{3k}\bar{x}_k$$

where

$$c_{ik} = b_{ir}a_{rj}B_{kj}, \qquad i, k = 1, 2, 3,$$

and B_{kj} denotes the cofactor of b_{kj} in $|b|$.

Let (a), (b), (c), (B) denote the matrices of the a_{ij}, b_{ij}, c_{ij}, B_{ij} respectively, and let (1) denote the unit matrix. Then the characteristic matrix $\bar{M}(t)$ of the new equations of the collineation is given by

$$\begin{aligned}\bar{M}(t) &= (c) - t(1)\\ &= (b)(a)(B)' - t(1)\\ &= (b)(a)(B)' - \bar{t}(b)(B)' \qquad \bar{t} = t|b|^{-1}\\ &= (b)(a)(B)' - \bar{t}(b)(1)(B)'\\ &= (b)[(a) - \bar{t}(1)](B)'\\ &= (b)M(\bar{t})(B)',\end{aligned}$$

$(B)'$ denoting the transpose of (B).

Since the determinant of a product of matrices is equal to the product of the determinants of the matrices, we have now

$$\bar{\Delta}(t) = |b|^3\Delta(\bar{t}).$$

The roots of the equation in t $\bar{\Delta}(t) = 0$ are thus the same as the roots of the equation in $\bar{t}$ $\Delta(\bar{t}) = 0$; hence, since $|b|\bar{t} = t$, the roots of $\bar{\Delta}(t) = 0$ are proportional to the roots of $\Delta(t) = 0$.

If a matrix is multiplied on either side by a non-singular matrix, the rank of the product matrix is equal to the rank of the given matrix. Since (b) and $(B)'$ are both non-singular, it follows that $\bar{M}(t)$ and $M(\bar{t})$ both have the same rank.

Hence, if τ is a root of $\Delta(t) = 0$ and $M(\tau)$ has rank n, $\tau|b|$ is a root of $\bar{\Delta}(t) = 0$ and $\bar{M}(\tau|b|)$ has the same rank n.

This result may also be obtained directly as follows. If the rank of $M(\tau)$ is n, the fundamental equations

$$\tau x_i = a_{ij}x_j$$

are equivalent to just n linearly independent equations. Changing to the new co-ordinate system, these equations become

$$\tau B_{ki}\bar{x}_k = a_{ij}B_{rj}\bar{x}_r.$$

Taking linear combinations of the last equations, we obtain the set

$$\tau b_{hi}B_{ki}\bar{x}_k = b_{hi}a_{ij}B_{rj}\bar{x}_r,$$

of which we can say only that they are equivalent to not more than n linearly independent equations. Now

$$b_{hi}B_{ki} = 0 \text{ if } h \neq k,$$
$$b_{hi}B_{ki} = |b| \text{ if } h = k,$$

and $$b_{hi}a_{ij}B_{rj} = c_{hr};$$

so the last set of equations may be written as

$$\tau|b|\bar{x}_h = c_{hr}\bar{x}_r.$$

Hence, the rank of $\bar{M}(\tau|b|)$ does not exceed n, the rank of $M(\tau)$. Moreover, we infer that the invariant points given by the equations $\tau x_i = a_{ij}x_j$ are among those given by the equations $\tau|b|\bar{x}_h = c_{hr}\bar{x}_r$

By considering the inverse change from the new co-ordinate system to the old, we prove in the same way that the rank of $M(\tau)$ does not exceed the rank of $\bar{M}(\tau|b|)$. Therefore the two ranks are equal. And, similarly, the two sets of invariant points given by the two sets of fundamental equations are identical.

(iii) Projective invariants of a collineation in one plane.—The algebra of part (ii) may be interpreted in another way. The equations there which represent the change from the old co-ordinate system to the new may be taken to represent a collineation of the given plane into itself or into another plane. The equations

$$\bar{x}_1' : \bar{x}_2' : \bar{x}_3' = c_{1k}\bar{x}_k : c_{2k}\bar{x}_k : c_{3k}\bar{x}_k$$

then represent a new collineation which we describe as a *projective transformation* of the old one.

Then what the algebra proves is that the ratios of the roots of the characteristic equation and the ranks of the characteristic matrix corresponding to these roots are invariant under the projective transformation These invariants are called (algebraic) *projective invariants* of the collineation.

(iv) Special and non-special collineations.—It is an easy matter to show that the derivative with respect to t of every minor of $\Delta(t)$ is expressible as a sum of minors of $\Delta(t)$ of order one less than that of the selected minor. For example, we have

$$-\frac{\partial\Delta(t)}{\partial t} =$$

$$\begin{vmatrix} a_{22}-t & a_{23} \\ a_{32} & a_{33}-t \end{vmatrix} + \begin{vmatrix} a_{33}-t & a_{31} \\ a_{13} & a_{11}-t \end{vmatrix} + \begin{vmatrix} a_{11}-t & a_{12} \\ a_{21} & a_{22}-t \end{vmatrix}$$

and

$$-\frac{\partial}{\partial t}\begin{vmatrix} a_{22}-t & a_{23} \\ a_{32} & a_{33}-t \end{vmatrix} = (a_{22}-t)+(a_{33}-t).$$

It follows that, if τ is a root of $\Delta(t)=0$ and if $M(\tau)$ has rank n, then τ is a root of multiplicity $\geqslant 3-n$, equality holding if $n=0$.

We call $h=3-n$ the *virtual multiplicity* of τ; and we have

$$\Sigma h \leqslant 3,$$

the summation covering all the roots of $\Delta(t)=0$. If equality holds in this summation, we say that the collineation is *non-special*; if inequality holds, the collineation is *special*.

Ex. 1. The collineation $x_1' : x_2' : x_3' = ax_1 : bx_2 : cx_3$ is non-special; and the collineation $x_1' : x_2' : x_3' = ax_1 : bx_2 + cx_3 : bx_3$ is special.

Ex 2. By analogy with part (iv), define special and non-special projectivities on a line; and show that the non-special projectivities are those with distinct united points and that the special projectivities are the parabolic projectivities.

[It is straightforward (and the reader should do this) to classify collineations in a space of any dimension k into non-special and special collineations, by direct analogy with the considerations in part (iv). With similar notation, the non-special collineations are then those for which $\Sigma h = k+1$ and the special collineations are those for which $\Sigma h < k+1$.]

Ex. 3. It was remarked in *Ex.* 15 of section 9 that a cyclic projectivity on a line cannot be parabolic. Deduce that a cyclic collineation in a plane is non-special, and show that the roots of its characteristic equation are proportional to nth roots of unity, n being the period of the collineation.

(v) The geometrical distinction between special and non-special collineations.—If τ is a root of $\Delta(t)=0$ for which $M(t)$ has rank n, the corresponding fundamental equations are equivalent to just n linearly independent linear equations; therefore they determine a set of invariant points which constitute a totally invariant space of dimension $2-n$ $(=h-1)$.

These totally invariant spaces, derived from all the distinct roots of $\Delta(t) = 0$, may all be contained by a line (as in the case $x_1' : x_2' : x_3' = ax_1 : bx_2 + cx_3 : bx_3$) or consist of one line (as in the case $x_1' : x_2' : x_3' = x_1 + ax_3 : x_2 : x_3$) or consist of one point (as in the case $x_1' : x_2' : x_3' = ax_1 : bx_1 + ax_2 + cx_3 : dx_1 + ax_3$). Accordingly, let k ($\leqslant 2$) be the dimension of the space of least dimension which contains all the totally invariant spaces mentioned. Then, as may easily be seen by enumerating all the possible cases,

$$\Sigma h \geqslant k + 1.$$

(This result also follows simply from a theorem quoted in part (viii) below.)

The space of dimension k, being determined by totally invariant spaces, is itself at least simply invariant under the collineation; and therefore the given collineation, in so far as it transforms the points of this space, induces a collineation (if $k = 2$) or projectivity (if $k = 1$) or identity (if $k = 0$) within the space. In regard to this induced transformation, the invariant spaces are just those of the given collineation. Hence, by applying the analogue, for a space of dimension k, of the inequality in part (iv),

$$\Sigma h \leqslant k + 1.$$

Therefore

$$\Sigma h = k + 1.$$

Thus, in the case of a non-special collineation, the least space which contains the totally invariant spaces is the plane itself. And, in the case of a special collineation, the totally invariant spaces are contained by a line or consist of one point.

(vi) Enumeration of the different types of collineation.—We are now ready to enumerate the various types of collineation which occur, beginning with those which are non-special.

Non-special collineations.

(*a*) $\Delta(t)$ has three different zeros τ_1, τ_2, τ_3. Each of these is necessarily a simple zero, making $M(t)$ have rank 2, and so $\Sigma h_i = 3$. Each zero gives rise to a single invariant point; and these points form a triangle, since it takes a plane to contain them. Taking a new triangle of reference with vertices at the invariant points, and using the fact that the ratios of τ_1, τ_2, τ_3 and the corresponding ranks of $M(t)$ are invariant, the equations of the collineation are seen to take the form

$$x_1' : x_2' : x_3' = \tau_1 x_1 : \tau_2 x_2 : \tau_3 x_3.$$

(*b*) $\Delta(t)$ has a simple zero τ_1 and a double zero τ_2. $M(\tau_1)$ necessarily has rank 2; but $M(\tau_2)$ may have rank 1 or 2. The

non-special case is where $M(\tau_2)$ has rank 1. Then τ_1 leads to a single invariant point and τ_2 to a totally invariant line, which does not contain the point, since the two need a plane to contain them. If we take the point to have co-ordinates (1, 0, 0) and the line to have the equation $x_1 = 0$, the equations of the collineation become

$$x_1' : x_2' : x_3' = \tau_1 x_1 : \tau_2 x_2 : \tau_2 x_3.$$

This is the case of a homology; it is harmonic if and only if $\tau_1 + \tau_2 = 0$.

(c) $\Delta(t)$ has a triple zero τ_1. $M(\tau_1)$ may have rank 2, 1 or 0, the non-special case is where $M(\tau_1)$ has rank 0. Then the fundamental equations are satisfied identically and every point is invariant. The collineation is identity; and its equations, referred to any triangle, are of the form

$$x_1' : x_2' : x_3' = x_1 : x_2 : x_3.$$

Special collineations.

(*d*) $\Delta(t)$ has a simple zero τ_1 and a double zero τ_2, for both of which $M(t)$ has rank 2. Here $h_1 + h_2 = 2$, there are just two distinct invariant points, but with different properties Let I_1 (1, 0, 0) arise from τ_1 and let I_2 (0, 1, 0) arise from τ_2. Then, since these are invariant points, the equations of the collineation have the form

$$x_1' : x_2' : x_3' = \tau_1 x_1 + a_{13} x_3 : \tau_2 x_2 + a_{23} x_3 : a_{33} x_3.$$

Since τ_2 is a double root of $\Delta(t) = 0$, $a_{33} = \tau_2$. The orientation of the line $x_1 = 0$ is at our disposal; let us choose it so that $a_{13} = 0$; then $x_1 = 0$ is a simply invariant line through I_2. The equations are then

$$x_1' : x_2' : x_3' = \tau_1 x_1 : \tau_2 x_2 + a_{23} x_3 . \tau_2 x_3.$$

This we recognise as the case where there are two infinitely near invariant points at I_2.

(*e*) $\Delta(t)$ has a triple zero τ_1 and $M(\tau_1)$ has rank 1. Here $h_1 = 2$; there is a totally invariant line which we take to be $x_1 = 0$. Since each point on the line is invariant, the equations of the collineation have the form

$$x_1' : x_2' : x_3' = \tau_1 x_1 : a_{21} x_1 + \tau_1 x_2 : a_{31} x_1 + \tau_1 x_3.$$

The collineation transforms the line (l_i) into the line (l_i'), where

$$l_1 : l_2 : l_3 = \tau_1 l_1' + a_{21} l_2' + a_{31} l_3' : \tau_1 l_2' : \tau_1 l_3'.$$

Corresponding lines meet on $x_1 = 0$; and, from the last equations,

they coincide when they pass through the point $(0, a_{21}, a_{31})$. We choose the triangle of reference more specifically now so that this point is $(0, 1, 0)$, that is so that $a_{31} = 0$. Then the equations of the collineation are

$$x_1' : x_2' : x_3' = \tau_1 x_1 : a_{21}x_1 + \tau_1 x_2 : \tau_1 x_3.$$

These are the equations of an elation with centre at $(0, 1, 0)$ and axis $x_1 = 0$.

(f) $\Delta(t)$ has a triple zero τ_1 and $M(\tau_1)$ has rank 2. There is a single invariant point I_1, which we take to be $(1, 0, 0)$. The equations of the collineation then have the form

$$x_1' : x_2' : x_3' = \tau_1 x_1 + a_{12}x_2 + a_{13}x_3 : \tau_1 x_2 + a_{23}x_3 : a_{32}x_2 + \tau_1 x_3.$$

We may further choose $x_3 = 0$ to be a simply invariant line through I_1; then $a_{32} = 0$. The equations become

$$x_1' : x_2' : x_3' = \tau_1 x_1 + a_{12}x_2 + a_{13}x_3 : \tau_1 x_2 + a_{23}x_3 : \tau_1 x_3.$$

These are the equations of a collineation which, from a geometrical point of view, we regard as having three infinitely near invariant points.

(vii) Projective equivalence of collineations.—We are now in a position to prove that it is sufficient for the projective equivalence of two collineations that they have the same projective invariants.

Two such collineations can, in fact, be transformed by separate collineations to the same standard form, which must be one of those enumerated in part (vi). We have, therefore, only to show that any two collineations in the same standard form are projectively equivalent.

It is enough to discuss one standard form as an example; and it may be left to the reader to deal similarly with the other forms. Type (f) is perhaps the least obvious case, so we fix attention on that.

We have then to consider two collineations, given by the two sets of equations

$$x_1' : x_2' : x_3' = \tau_1 x_1 + a_{12}x_2 + a_{13}x_3 : \tau_1 x_2 + a_{23}x_3 : \tau_1 x_3,$$
$$y_1' : y_2' : y_3' = \tau_1 y_1 + b_{12}y_2 + b_{13}y_3 : \tau_1 y_2 + b_{23}y_3 : \tau_1 y_3.$$

A collineation which transforms one set of equations into the other must transform the invariant point $x_1 : x_2 : x_3 = 1 : 0 : 0$ and the invariant line $x_3 = 0$ into the invariant point $y_1 : y_2 : y_3 = 1 : 0 : 0$ and the invariant line $y_3 = 0$; and must therefore have the form

$$x_1 : x_2 : x_3 = py_1 + qy_2 + ry_3 : vy_2 + wy_3 : y_3.$$

It may now be verified by direct computation that the two given sets of equations are equivalent under the collineation for which

$$p = \frac{a_{12}a_{23}}{b_{12}b_{23}}, \quad q = 0, \quad r = 0,$$

$$v = \frac{a_{23}}{b_{23}}, \quad w = \frac{b_{13}a_{23}}{b_{12}b_{23}} - \frac{a_{13}}{a_{12}}.$$

(viii) Comments.—It will have been noticed that the algebraic treatment leads to a grouping of the different types of collineation which differs from that arising from the geometrical approach. The collineations which we described earlier as *general* are characterised algebraically by the fact that every root of $\Delta(t) = 0$ makes $M(t)$ have rank 2. In this connection we should note that, in general, a multiple root of $\Delta(t) = 0$ does not necessarily annul all the second order minors of $\Delta(t)$.

Then, also, the geometrical treatment leads to the idea of infinitely near invariant points. To reach this idea algebraically, we may choose to regard cases (*d*), (*f*) as limiting cases of (*a*). The reader should have no difficulty in carrying out the limiting processes.

The language of this section has been in as general terms as possible. This has been done with a view to suggesting the general treatment of collineations in a space of any number of dimensions. This generality is especially apparent in part (v), where it seemed preferable to give expression to a geometrical theorem rather than to enumerate a series of algebraic details. In that part, we have used the theorem that the linear space of least dimension k, which contains a given set of linear spaces of dimensions $n_1, n_2, \ldots, n_r$, has $k + 1 \leqslant n_1 + n_2 + \ldots + n_r + r$. This theorem is very easily verified in the cases $k = 1, 2$, which are all that we need here

53. Correlations.

(i) Definitions of correlation and attached collineation.—It is natural to consider, after collineations, a somewhat similar type of correspondence between the points of one plane π and the lines of another, or coincident, plane π'.

Let there be a (1, 1) correspondence between the points P of π and the lines P' of π'. The correspondence determines a *direct transformation* K which transforms P into P' and an *inverse transformation* K^{-1} which transforms P' into P. The relation between P, P' is symbolised by the notation $\mathsf{P}' = K(P)$, $P = K^{-1}(\mathsf{P}')$.

If the points of every line P in π transform into the lines through a point P' in π', and *vice versa*, K is called a *correlation*; and it follows that K^{-1} is also a correlation.

If, in all cases, the correspondence between the points of a line in one plane and the lines through a point in the other plane is projective, we say that the correlation is a *projective correlation*. In this section we are concerned only with projective correlations; so from now on we take "correlation" to mean "projective correlation."

An obvious example of a correlation is reciprocation with respect to an irreducible conic; but, as we shall see, this is not the most general kind of correlation.

In the case of coincident planes π, π', there is a collineation determined by the relations

$$P' = K(\mathsf{P}),\ \mathsf{P} = \mathsf{P}',\ \mathsf{P}' = K(P).$$

This collineation, transforming P into P', is called the *attached collineation* of K; denoting it by T, an obvious definition of the product of two correlations gives $T = K^2$.

(ii) Fundamental theorem.—Collineations and correlations may be expected to have a number of analogous properties. Thus we should expect the following fundamental theorem to be true, as it is: there is a unique correlation which transforms any given quadrangle $ABCD$ of π into any given quadrilateral $\mathsf{A'B'C'D'}$ of π'.

Take any irreducible conic K' in π', and let A', B', C', D' be the poles of A', B', C', D' with respect to K'. Let S be the collineation which transforms $ABCD$ into $A'B'C'D'$ and L the correlation which transforms points of π' into their polars with respect to K'. Then the transformation S followed by the transformation L determines a correlation, denoted by LS, which transforms $ABCD$ into $\mathsf{A'B'C'D'}$.

The uniqueness of the correlation is proved as in section 49 (i)

(iii) Equations of a correlation.—Let (l_i') be co-ordinates of P' and (x_i) be co-ordinates of P. Then, as in section 49 (ii), we may easily show that the correlation K is represented by equations of the form

$$\rho l_i' = a_{ij}x_j, \quad i = 1, 2, 3,$$

with $\rho \neq 0$, $|a| \neq 0$.

The equations of K^{-1} are then

$$\sigma l_i = a_{ji}x_j', \quad i = 1, 2, 3.$$

If the planes π, π' coincide, we may assume without loss of generality that all co-ordinates refer to a common system of reference. The equations of the attached collineation T then become

$$\theta a_{ji}x_j' = a_{ij}x_j, \quad i = 1, 2, 3.$$

(iv) The incidence conic and conic-envelope of a correlation in one plane.—We now restrict attention to correlations in one plane. Corresponding to the notion of invariant point in a collineation, we have that of *incident pair* in a correlation. A point P and its corresponding line P′ are said to constitute an incident pair when P is on P′.

For such a pair we have

$$l_i'x_i = 0,$$

and therefore

$$a_{ij}x_ix_j = 0,$$

showing that the points P lie on a conic K, called the *incidence conic*; conversely, if P is on K, it belongs to an incident pair. Similarly, the lines of the incident pairs are the lines of a conic-envelope K′, called the *incidence conic-envelope*, whose equation is

$$A_{ij}l_i'l_j' = 0,$$

where A_{ij} is the cofactor of a_{ij} in $|a|$. In general, K and K′ are not associated.

Ex. 1. P is a point on K, prove that the lines of K′ which pass through P are correlative to P in K, K^{-1}.

(v) Involutory pairs of elements.—A point P and its correlative line P′ are said to be an *involutory pair* when $P = K$ (P′). Then the co-ordinates of P satisfy the equations

$$a_{ij}x_j = \tau a_{ji}x_j, \quad i = 1, 2, 3,$$

where τ is a zero of the determinant

$$D(t) \equiv \begin{vmatrix} a_{11} - ta_{11} & a_{12} - ta_{21} & a_{13} - ta_{31} \\ a_{21} - ta_{12} & a_{22} - ta_{22} & a_{23} - ta_{32} \\ a_{31} - ta_{13} & a_{32} - ta_{23} & a_{33} - ta_{33} \end{vmatrix}.$$

The involutory points P are, in fact, the invariant points of the attached collineation T; and the involutory lines are the invariant lines of T. It appears to be natural to classify K on the basis of the algebraic system of classification already adopted for collineations.

$D(1)$ is a skew-symmetric determinant of odd order and therefore is zero, as may be verified directly. Thus 1 is always a root of $D(t) = 0$; the other two roots have the form τ, τ^{-1}.

Ex. 2. The involutory points corresponding to a root $\tau\ (\neq 1)$ of $D(t) = 0$ lie on the incidence conic K; and the correlative involutory lines belong to K′.

(vi) The projective invariants of the attached collineation.—If the equations of T are expressed in the form

$$\theta x_i' = c_{ij}x_j,$$

we have

$$a_{ij} = a_{ri}c_{rj}$$

and therefore, with the usual matrix notation,

$$\begin{pmatrix} a_{11} - ta_{11} & a_{12} - ta_{21} & a_{13} - ta_{31} \\ a_{21} - ta_{12} & a_{22} - ta_{22} & a_{23} - ta_{32} \\ a_{31} - ta_{13} & a_{32} - ta_{23} & a_{33} - ta_{33} \end{pmatrix}$$

$$\begin{aligned} &= (a) - t(a)' \\ &= (a)'(c) - t(a)'(1) \\ &= (a)'[(c) - t(1)] \\ &= \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} c_{11} - t & c_{12} & c_{13} \\ c_{21} & c_{22} - t & c_{23} \\ c_{31} & c_{32} & c_{33} - t \end{pmatrix}. \end{aligned}$$

Hence the characteristic determinant of the attached collineation T is a constant multiple of $D(t)$, and, for equal values of t, the ranks of the corresponding matrices are equal. Let us denote by $L(t)$ the matrix corresponding to $D(t)$, that is

$$L(t) \equiv (a) - t(a)'.$$

The ratios of the roots of $D(t) = 0$ and the ranks of $L(t)$ which correspond to these roots are projective invariants of T.

(vii) Classification of correlations having non-special attached collineations.—We consider now the particular types of correlation which arise, beginning with the cases where T is non-special.

(*a*) $D(t)$ has zeros 1, τ, τ^{-1}, with $\tau^2 \neq 1$. The corresponding ranks of $L(t)$ are each 2 There are three involutory points, not in line; and, with these as vertices of the triangle of reference, the equations of T may be taken in the form

$$x_1' : x_2' : x_3' = x_1 : \tau x_2 : \tau^{-1}x_3.$$

From the identity

$$(a) = (a)'(c),$$

we find that

$$a_{12} = a_{21} = a_{13} = a_{31} = a_{22} = a_{33} = 0,\ a_{32} = \tau a_{23}.$$

The equations of the correlation therefore have the form

$$l_1' : l_2' . l_3' = a_{11}x_1 : a_{23}x_3 : \tau a_{23}x_2,$$

with $a_{11}a_{23} \neq 0$.

The incidence conic K has the equation

$$a_{11}x_1^2 + a_{23}(1 + \tau)x_2x_3 = 0$$

and it touches the involutory lines $x_3 = 0$, $x_2 = 0$ at the involutory points arising from τ, τ^{-1} respectively; and it is to be observed that K does not contain the involutory point (1, 0, 0) arising from the unit root of $D(t) = 0$.

The incidence conic-envelope K′ has the equation

$$\tau a_{23} l_1'^2 + (1 + \tau) a_{11} l_2' l_3' = 0$$

and it contains the involutory lines $x_2 = 0$, $x_3 = 0$ but not $x_1 = 0$. Its associated conic touches K at the involutory points arising from τ, τ^{-1}.

(*b*) $D(t)$ has zeros 1, -1, -1; $L(1)$ has rank 2 and $L(-1)$ has rank 1. T is a harmonic homology whose equations may be taken to be

$$x_1' : x_2' : x_3' = x_1 : -x_2 : -x_3.$$

We find, again from the identity $(a) = (a)'(c)$, that

$$a_{12} = a_{21} = a_{13} = a_{31} = a_{22} = a_{33} = a_{23} + a_{32} = 0.$$

Therefore K has equations of the form

$$l_1' : l_2' : l_3' = a_{11}x_1 : a_{23}x_3 : -a_{23}x_2.$$

The equation of K is $x_1^2 = 0$, which is that of the axis of homology counted twice, and the equation of K′ is $l_1'^2 = 0$, which is that of the pencil, with vertex at the centre of homology, counted twice.

(*c*) $D(t)$ has zeros 1, 1, 1 and $L(1)$ has rank 0. In this case $a_{ij} = a_{ji}$ for all i, j. The matrix (a) is symmetric; and the equations of the correlation express that a point P and its correlative line P′ are respectively pole and polar relative to the irreducible incidence conic K. The conic-envelope K′ is now associated with K. We have $K = K^{-1}$, $T = I$; K is called a *polarity*.

Ex. 3. A correlation in which the sides of a triangle are correlative with the opposite vertices is a polarity.

Ex. 4. Prove that K, K′ are associated if and only if K is a polarity.

Ex. 5. Any collineation may be expressed as a product of two polarities. (*Cf.* section 9 (vi) (*d*).)

(viii) Classification of correlations having special attached collineations.—We come now to the cases where T is special.

(*d*) $D(t)$ has zeros 1, -1, -1; $L(1)$, $L(-1)$ both have rank

2. T has a simple invariant point and two other infinitely near invariant points; we take the equations of T in the form,

$$x_1' : x_2' : x_3' = x_1 : -x_2 + bx_3 : -x_3.$$

We find that

$$a_{12} = a_{21} = a_{22} = a_{13} = a_{31} = 0,\ a_{32} = -a_{23},\ 2a_{33} = ba_{23}.$$

The equations of K therefore have the form

$$l_1' : l_2' : l_3' = a_{11}x_1 : a_{23}x_3 \,.\, -a_{23}x_2 + \tfrac{1}{2}ba_{23}x_3.$$

The equation of K is

$$a_{11}x_1^2 + \tfrac{1}{2}ba_{23}x_3^2 = 0,$$

so that K consists of two lines harmonic with regard to the involutory lines $x_1 = 0$, $x_3 = 0$. And the equation of K′ is

$$a_{23}l_1'^2 + \tfrac{1}{2}ba_{11}l_2'^2 = 0,$$

so that K′ consists of two pencils with vertices harmonic with regard to the involutory points (1, 0, 0), (0, 1, 0).

(e) $D(t)$ has zeros 1, 1, 1; and $L(1)$ has rank 1. T is an elation whose equations may be taken to be

$$x_1' : x_2' : x_3' = x_1 : bx_1 + x_2 : x_3,$$

the axis of the elation being $x_1 = 0$ and the centre (0, 1, 0). We find that

$$a_{12} = a_{21} = a_{22} = a_{23} = a_{32} = 0, \quad a_{13} = a_{31}.$$

These equalities however require $|a| = 0$. Therefore there is no correlation whose attached collineation is an elation.

(f) $D(t)$ has zeros 1, 1, 1, $L(1)$ has rank 2. T has three infinitely near invariant points; its equations may be taken as

$$x_1' : x_2' : x_3' = x_1 + bx_2 + cx_3 : x_2 + dx_3 : x_3.$$

We find that

$$a_{11} = a_{12} = a_{21} = 0, \quad a_{13} = a_{31}, \quad a_{23} - a_{32} = da_{22} = -ba_{13},$$
$$ca_{13} + da_{23} = 0.$$

The equations of K therefore have the form

$$l_1' : l_2' : l_3' = da_{13}x_1 : -a_{13}(bx_2 + cx_3) : a_{13}(dx_1 + (bd - c)x_2) + a_{33}dx_3.$$

The equation of K is

$$a_{13}(-bx_2^2 + (bd - 2c)x_2x_3 + 2dx_3x_1) + a_{33}dx_3^2 = 0.$$

Thus K is an irreducible conic touching the involutory line at the involutory point.

54. Projective invariants.

(i) Absolute projective invariants.—Any numerical or geometrical property of a configuration of points, lines, curves in a plane which is unchanged when the configuration is subjected to any collineation is called a *projective invariant* of the configuration. In order to distinguish these projective invariants from other semi-invariants, to be mentioned shortly, the description *absolute* is applied.

Two configurations are said to be *projectively equivalent* when either may be transformed into the other by means of a collineation; they have the same projective invariants.

For example, any two irreducible conics are projectively equivalent (the proof is indicated in *Ex.* 7 of section 51). In fact, also, any two reducible conics, each consisting of a pair of distinct lines or each a line counted twice, are projectively equivalent.

In this book, attention has been concentrated on projectively invariant properties of configurations. It is therefore natural to enquire into the classification of configurations according to their numerical projective invariants. This process has already been carried out in some cases; and now we consider further aspects of the matter.

(ii) Relative projective invariants.—If the collineation

$$x_i' = c_{ij}x_j$$

transforms the homogeneous quadratic polynomial $a_{ij}x_ix_j$ (with $a_{ij} = a_{ji}$) into $a_{ij}'x_i'x_j'$ (with $a_{ij}' = a_{ji}'$), we have

$$|a| = |c|^2|a'|.$$

We call $|c|$ the *modulus* of the collineation. Since $|a|$ differs from $|a'|$ only by a factor which is a power of the modulus, we say that $|a|$ is a *relative projective invariant* of the conic $a_{ij}x_ix_j = 0$, in distinction from an absolute invariant which does not involve the modulus.

The same algebra shows that $|a|$ is a relative invariant with respect to any change of the system of reference.

(iii) Projectively equivalent pairs of conics.

(*a*) We consider next the circumstances in which two pairs of conics are projectively equivalent.

Let the conics A, B, given respectively by the equations

$$A \equiv a_{ij}x_ix_j = 0, \quad a_{ij} = a_{ji} \text{ for all } i, j,$$
$$B \equiv b_{ij}x_ix_j = 0, \quad b_{ij} = b_{ji} \text{ for all } i, j,$$

be transformed by the collineation

$$x_r' = c_{rs}x_s, \quad |c| \neq 0.$$

They become the conics A′, B′, given by

$$A' \equiv a_{ij}'x_i'x_j' = 0, \quad B' \equiv b_{ij}'x_i'x_j' = 0,$$

where

$$a_{ij} = c_{ri}a_{rs}'c_{sj}, \quad b_{ij} = c_{ri}b_{rs}'c_{sj}.$$

The *characteristic matrix* of the two polynomials A, B is

$$\begin{aligned} M(\lambda) \equiv & \begin{pmatrix} a_{11} + \lambda b_{11} & a_{12} + \lambda b_{12} & a_{13} + \lambda b_{13} \\ a_{21} + \lambda b_{21} & a_{22} + \lambda b_{22} & a_{23} + \lambda b_{23} \\ a_{31} + \lambda b_{31} & a_{32} + \lambda b_{32} & a_{33} + \lambda b_{33} \end{pmatrix} \\ \equiv & (a) + \lambda(b); \end{aligned}$$

and we define the *characteristic determinant* $D(\lambda)$ of the two polynomials to be the determinant of this matrix. The equation

$$D(\lambda) \equiv |(a) + \lambda(b)| = 0$$

is called the *characteristic equation* or *discriminating cubic* of A and B; its roots determine those members of the pencil determined by A, B which are reducible.

Let $M'(\lambda)$ be the characteristic matrix of A' and B'; then, if a *bar* now denotes a transposed matrix, we have

$$\begin{aligned} M(\lambda) &= (a) + \lambda(b) \\ &= (\bar{c})(a')(c) + \lambda(\bar{c})(b')(c) \\ &= (\bar{c})[(a') + \lambda(b')](c) \\ &= (\bar{c})M'(\lambda)(c). \end{aligned}$$

Hence, if $D'(\lambda)$ is the characteristic determinant of A', B', we have

$$D(\lambda) = |c|^2 D'(\lambda).$$

Since A, B are unaltered if A, B are multiplied independently by any non-zero constants, it follows that the ratios of the roots of $D(\lambda) = 0$ are absolute projective invariants of the pair A, B By an argument just like that of section 52 (ii), the ranks of $M(\lambda)$, for those values of λ given by $D(\lambda) = 0$, are also absolute projective invariants of the pair A, B. These two sets of invariants are the *fundamental invariants* of the pair of conics.

It is to be observed that the algebra proves also that the fundamental invariants of the pair A, B are absolutely invariant with respect to a change of the system of reference.

(*b*) We now prove that, conversely, two pairs of conics with the same fundamental invariants are projectively equivalent. For brevity, we limit the discussion to the case where all the conics are irreducible, though there is no difficulty in dealing with the cases where two or all of the conics may be reducible.

(α) $D(\lambda) = 0$ has three different roots λ_1, λ_2, λ_3. $M(\lambda_i)$ necessarily has rank 2. The pencil of conics determined by A, B has three distinct line-pairs; and therefore A, B have four distinct common points. With the diagonal triangle of this set of points as triangle of reference, we may take $A \equiv \alpha_i x_i^2$, $B \equiv \beta_i x_i^2$, with the ratios α_1/β_1, α_2/β_2, α_3/β_3 all different. These ratios are then the same as $-\lambda_1$, $-\lambda_2$, $-\lambda_3$ in some order which we may suppose, without loss of generality, to be the order written.

Similarly we may express A', B' in the forms $\alpha_i' x_i'^2$, $\beta_i' x_i'^2$ respectively, with similar remarks following.

Since the two sets of fundamental invariants are the same, we have

$$\frac{\alpha_1}{\beta_1} : \frac{\alpha_2}{\beta_2} : \frac{\alpha_3}{\beta_3} = \frac{\alpha_1'}{\beta_1'} : \frac{\alpha_2'}{\beta_2'} : \frac{\alpha_3'}{\beta_3'}.$$

From this it is clear that the collineation

$$x_1 = \left(\frac{\alpha_1'}{\alpha_1}\right)^{\frac{1}{2}} x_1', \quad x_2 = \left(\frac{\alpha_2'}{\alpha_2}\right)^{\frac{1}{2}} x_2', \quad x_3 = \left(\frac{\alpha_3'}{\alpha_3}\right)^{\frac{1}{2}} x_3$$

transforms the conics A, B into the conics A′, B′ respectively.

(β) $D(\lambda) = 0$ has roots λ_1, λ_1, λ_2, with $\lambda_1 \neq \lambda_2$. $M(\lambda_2)$ necessarily has rank 2 but $M(\lambda_1)$ may have rank 1 or 2.

In both cases the pencil of conics contains only two distinct line-pairs A, B therefore touch at one or two points.

Suppose first that A, B touch at P and meet again at two points Q, R. Take PQR as triangle of reference and choose the unit point so that the common tangent at P is given by $x_2 + x_3 = 0$. Then we may take

$$A \equiv 2x_1(x_2 + x_3) + 2\alpha x_2 x_3, \quad B \equiv 2x_1(x_2 + x_3) + 2\beta x_2 x_3,$$

with $\alpha \neq \beta$. Hence, $\lambda_1 = -1$, $\lambda_2 = -\alpha/\beta$. This is the case where $M(\lambda_1)$ has rank 2; we shall see in a moment that, when A, B have double contact, $M(\lambda_1)$ has rank 1.

In the same way, then, A′, B′ touch at one point only and we may take

$$A' \equiv 2x_1'(x_2' + x_3') + 2\alpha' x_2' x_3', \quad B' \equiv 2x_1'(x_2' + x_3') + 2\beta' x_2' x_3',$$

with $\alpha' \neq \beta'$. By the invariance of λ_1/λ_2, $\alpha/\beta = \alpha'/\beta'$. Hence the collineation

$$\alpha' x_1 = \alpha x_1', \; x_2 = x_2', \; x_3 = x_3'$$

transforms A, B into A′, B′. (This collineation is a homology with centre P and axis QR if the frames of reference coincide.)

Now suppose that A, B touch at Q, R, their common tangents meeting at P. With PQR as triangle of reference, we may take

$$A \equiv \alpha x_1^2 + 2x_2x_3, \quad B \equiv \beta x_1^2 + 2x_2x_3,$$

with $\alpha \neq \beta$. Then $\lambda_1 = -1$, $\lambda_2 = -\alpha/\beta$, and $M(\lambda_1)$ has rank 1.

Similarly, we may take

$$A' \equiv \alpha' x_1'^2 + 2x_2'x_3', \quad B' \equiv \beta' x_1'^2 + 2x_2'x_3',$$

with $\alpha' \neq \beta'$. By the invariance of λ_1/λ_2, $\alpha/\beta = \alpha'/\beta'$. Therefore A, B correspond to A′, B′ in the collineation

$$x_1 = \left(\frac{\alpha'}{\alpha}\right)^{\frac{1}{2}} x_1', \quad x_2 = x_2', \quad x_3 = x_3'.$$

(This collineation is a homology with centre P and axis QR if the frames of reference coincide.)

(γ) $D(\lambda) = 0$ has roots $\lambda_1, \lambda_1, \lambda_1$. $M(\lambda_1)$ may have rank 1 or 2.

In both cases the pencil of conics has only one distinct line-pair. A, B therefore have three-point contact at a point P and meet again at one more point Q; or else they have four-point contact.

Suppose first that A, B have three-point contact. Let the tangents at P, Q to A meet at R. Then, with PQR as triangle of reference and unit point on A, we may take

$$A \equiv 2x_3^2 - 2x_1x_2, \quad B \equiv 2x_3^2 - 2x_1x_2 + 2\alpha x_2x_3.$$

Hence $\lambda_1 = -1$ and $M(\lambda_1)$ has rank 2. We shall see that, when A, B have four-point contact, $M(\lambda_1)$ has rank 1.

Similarly we may take

$$A' \equiv 2x_3'^2 - 2x_1'x_2', \quad B' \equiv 2x_3'^2 - 2x_1'x_2' + 2\alpha' x_2'x_3'.$$

This time no relation necessarily connects α, α'. The collineation

$$\alpha' x_1 = \alpha x_1', \quad \alpha x_2 = \alpha' x_2', \quad x_3 = x_3'$$

transforms A, B into A′, B′ respectively.

Lastly, if A, B have four-point contact at P, let the tangent to A at another point Q meet the tangent to both conics at P in R; take PQR as triangle of reference and the unit point on A. Then we may take

$$A \equiv 2x_3^2 - 2x_1x_2, \quad B \equiv 2x_3^2 - 2x_1x_2 + \alpha x_2^2.$$

We have $\lambda_1 = -1$ and $M(\lambda_1)$ has rank 1.

Similarly, we may take

$$A' \equiv 2x_3'^2 - 2x_1'x_2', \quad B' \equiv 2x_3'^2 - 2x_1'x_2' + \alpha' x_2'^2.$$

There is no relation between α, α'. The collineation

$$x_1 = \left(\frac{\alpha}{\alpha'}\right)^{\frac{1}{4}} x_1', \quad x_2 = \left(\frac{\alpha'}{\alpha}\right)^{\frac{1}{4}} x_2', \quad x_3 = x_3'$$

transforms A, B into A′, B′ respectively.

Ex 1. If A, B are both irreducible, there is an associated collineation T defined as follows: a point $P(x)$ has a polar line P relative to A, and Q (y) is the pole of P relative to B; P, Q are corresponding points. Prove that the equations of T are

$$a_{ij}x_j = \theta b_{ij}y_j, \quad i = 1, 2, 3.$$

A′, B′ are two other irreducible conics in the same plane as A, B or in another plane, and T' is the associated collineation Prove that, if A, B are projectively equivalent to A′, B′, then T is projectively equivalent to T'.

Ex. 2 A pencil of conics, with four distinct base points, is projectively equivalent to any other such pencil; and there are 24 collineations which effect the equivalence.

Ex 3. A is an irreducible conic and B is a reducible conic Prove that, except for at most three values of θ the conic $A + \theta B = 0$ is irreducible, and that, if $N(\mu)$ is the characteristic matrix of $A = 0$, $A + \theta B = 0$, θ not being one of these particular values (of which one is ∞), then

$$N(\mu) = M(\lambda) \begin{pmatrix} \mu + 1 & 0 & 0 \\ 0 & \mu + 1 & 0 \\ 0 & 0 & \mu + 1 \end{pmatrix},$$

where $\lambda(\mu + 1) = \mu\theta$.

Hence deduce, using part (iii) of this section, that if A′, B′ is another pair of conics, irreducible and reducible respectively, and having the same fundamental invariants as A, B, then A, B and A′, B′ are projectively equivalent pairs

The reasoning may be extended to the case of two pairs of reducible conics A, B and A′, B′, neither pair having a line in common · but in this case it is much more direct to consider a collineation which transforms the four common points of A, B into those of A′, B′.

(iv) Relative projective invariants of a pair of conics.—The fundamental invariants of A, B are of less frequent use than certain relative projective invariants which we next consider.

The polynomial $D(\lambda)$ may be written in the form

$$\Delta + \lambda\Theta + \lambda^2\Theta^* + \lambda^3\Delta^*,$$

where

$$\begin{aligned} \Delta &\equiv |a| = \tfrac{1}{3}a_{ij}A_{ij}, \\ \Theta &\equiv b_{ij}A_{ij}, \\ \Theta^* &\equiv a_{ij}B_{ij}, \\ \Delta^* &\equiv |b| = \tfrac{1}{3}b_{ij}B_{ij}, \end{aligned}$$

using the ordinary notation of cofactors (*cf* section 25 (iv)).

By the relation $D(\lambda) = |c|^2 D'(\lambda)$ of part (iii) of this section, Δ, Θ, Θ^*, Δ^* are relative projective invariants of the pair of polynomials A, B

Any ratio of two polynomials, homogeneous in Δ, Θ, Θ^*. Δ^*, which are also homogeneous separately in the a_{ij} and in the b_{ij} and are of the same degree in the a_{ij} as in the b_{ij}, is an absolute projective invariant of the two conics A, B Among the simplest of such absolute invariants are

$$\frac{\Theta^2}{\Delta\Theta^*}, \quad \frac{\Theta^{*2}}{\Delta^*\Theta}, \quad \frac{\Theta^3}{\Delta^2\Delta^*}, \quad \frac{\Theta^{*3}}{\Delta\Delta^{*2}};$$

the last two of these may each be expressed in terms of the first two.

Ex. 4 We may develop a theory of the projective equivalence of two pairs of conic-envelopes in exactly the same way as for two pairs of conics In the case of a pair of irreducible conic-envelopes $A_{ij}l_il_j = 0$, $B_{ij}l_il_j = 0$, prove that the corresponding relative invariants, having regard to the usual notation, may be expressed in the forms Δ^2, $\Delta\Theta^*$, $\Delta^*\Theta$, Δ^{*2} respectively.

(v) Geometrical significance of the vanishing of a relative invariant.—The relation $\Delta = 0$ implies that the conic A and all projectively equivalent conics are reducible. Similarly, the relation $\Theta = 0$ implies a geometrical projectively invariant relation between A, B which we investigate next.

Let us suppose first of all that A, B are both irreducible. We make use of the fact that the relation $\Theta = 0$ is invariant under any change of the system of reference We therefore choose a triangle of reference PQR as follows P is to be any point of B not on the reciprocal of B relative to A; the polar of P relative to A then meets B in two distinct points, one of which is taken to be Q, and with P not on the tangent to A at a common point of A and B the polars of P, Q relative to A meet in R, not on PQ

Then, PQR being self-polar relative to A, we may take $A \equiv a_1x_1^2 + a_2x_2^2 + a_3x_3^2$, with $a_1a_2a_3 \neq 0$, and $B \equiv bx_3^2 + 2b_1x_2x_3 + 2b_2x_3x_1 + 2b_3x_1x_2$, with $2b_1b_2b_3 \neq bb_3^2$. Hence $\Theta = ba_2a_3$, and therefore $b = 0$, so that R lies on B.

Thus $\Theta = 0$ implies that, in an infinity of ways, triangles may be inscribed in B which are self-polar relative to A; that is to say B is outpolar to A.

Conversely, if B is outpolar to A, then $\Theta = 0$. For then there exists a triangle in regard to which A, B may be expressed as above with $b = 0$.

Ex. 5. Prove, in a similar way, that $\Theta = 0$ is a necessary and sufficient condition for an infinity of triangles to exist, each of which is circumscribed about A and self-polar relative to B.

Ex. 6. If B consists of a pair of distinct lines, prove that $\Theta = 0$ is a necessary and sufficient condition for the lines to be conjugate relative to A.

Ex. 7. If B consists of a line counted twice, prove that $\Theta = 0$ is a necessary and sufficient condition for the line to touch A.

Ex. 8 A necessary and sufficient condition that there may exist an infinity of triangles inscribed in A and circumscribed about B, both conics being irreducible, is $\Theta^{*2} = 4\Delta^*\Theta$.

Ex. 9 If B is inpolar to A_1, A_2, then B is inpolar to every conic of the pencil determined by A_1, A_2.

Ex. 10. The co-ordinates (x) of any point on any line-pair in the pencil determined by A, B satisfy the equation $A + \lambda B = 0$ with $D(\lambda) = 0$; and conversely. Hence the equation representing all the line-pairs in the pencil is

$$\Delta B^3 - \Theta B^2 A + \Theta^* B A^2 - \Delta^* A^3 = 0.$$

Ex. 11. The equation representing all the point-pairs in the range determined by the conic-envelopes $\Sigma = 0$, $\Psi = 0$, associated with A, B, is

$$\Delta^2\Psi^3 - \Delta\Theta^*\Psi^2\Sigma + \Delta^*\Theta\Psi\Sigma^2 - \Delta^{*2}\Sigma^3 = 0.$$

Ex. 12. If A, B touch at one or at two points, two roots of $D(\lambda) = 0$ are equal, and therefore

$$(9\Delta\Delta^* - \Theta\Theta^*)^2 = 4(\Theta^2 - 3\Delta\Theta^*)(\Theta^{*2} - 3\Delta^*\Theta).$$

Ex. 13. If A, B have three- or four-point contact, all the roots of $D(\lambda) = 0$ are equal, and therefore

$$\frac{3\Delta}{\Theta} = \frac{\Theta}{\Theta^*} = \frac{\Theta^*}{3\Delta^*}.$$

(vi) Covariant curves.—A curve C, which is derived from two conics A, B, by any projective construction is projectively equivalent to the curve C′, derived by the same construction from a projectively equivalent pair of conics A′, B′. We say that C is a *covariant curve* of A, B.

Examples of covariant curves are (i) the harmonic conic of the conic-envelopes associated with A, B; (ii) the conic associated with the harmonic conic-envelope of A, B; (iii) the set of four common tangents of A, B.

Since a knowledge of the relative invariants of A, B amounts to a knowledge of some of the fundamental invariants of A, B, we might expect that some covariant curves would have their equations expressible in terms of

$$\Delta, \Theta, \Theta^*, \Delta^*, A, B;$$

and, corresponding to the influence of the remaining fundamental invariants, that other covariant curves might have their equations expressible in terms of these six expressions together with those expressions whose vanishing represent certain particular covariant curves. We had an example of the first circumstance in *Ex.* 10; other examples are to be found below.

Starting instead with two conic-envelopes Σ, Ψ, we may define, in the same way, *contra-variant* envelopes of Σ, Ψ; and make similar inferences.

Ex. 14. The equation of the set of four common tangents of A, B is $F^2 = 4\Delta\Delta^*AB$, where $F = 0$ is the harmonic conic of the conic-envelopes associated with A, B (the polynomial F is defined to be the polynomial denoted by K on page 178, allowing for differences of notation).

Ex. 15. The equation of the set of four common points of A, B is $\Phi^2 = 4\Sigma\Psi$, where $\Sigma = 0$, $\Psi = 0$ are the conic-envelopes associated with A, B and $\Phi = 0$ is the harmonic conic-envelope of A, B (the polynomial Φ is defined on page 179, apart from a difference of notation)

Ex 16. The conic associated with $\Phi = 0$ has the equation $F = \Theta^*A + \Theta B$; and the conic-envelope associated with $F = 0$ has the equation $\Delta\Delta^*\Phi = \Theta\Delta^*\Sigma + \Theta^*\Delta\Psi$.

Ex. 17. The conic-envelope which is the reciprocal of B relative to A has the equation $\Theta\Sigma = \Delta\Phi$.

Ex. 18. The conic which is the reciprocal of $\Psi = 0$ relative to A has the equation $F = \Theta^*A$.

Ex 19. The reciprocal, with regard to A, of the harmonic envelope of A and B has the equation $\Theta A = \Delta B$.

(vii) Similar conics.—(*a*) Two irreducible conics A, A′ are said to be *similar* when they are projectively equivalent under a similitude.

If then a similitude S transforms A into A′, either A, A′ are both central conics and S transforms the centre, foci, axes, directrices and director-circle of A into the centre, foci, axes, directrices and director-circle of A′ or else A, A′ are both parabolas and S transforms the focus, axis and directrix of A into the focus, axis and directrix of A′.

(*b*) Let $\Sigma = 0$, $\Sigma' = 0$ be the equations of the conic-envelopes associated with two conics A, A′; and let $I = 0$, $J = 0$ now represent the pencils of lines with vertices at the circular points. Then A, A′ are similar if and only if the pairs of conic-envelopes $\Sigma = 0$, $IJ = 0$ and $\Sigma' = 0$, $IJ = 0$ are projectively equivalent.

With modified complex co-ordinates based on rectangular axes, let

$$A \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$
$$IJ \equiv l^2 + m^2.$$

Then the relative invariants of the pair $\Sigma = 0$, $IJ = 0$ are

$$\Delta^2,\ \Delta(a+b),\ ab - h^2,\ 0.$$

Hence, the ratio $(ab - h^2)/(a+b)^2$ is an absolute invariant of A under the similarity group of collineations.

Ex 20. If A is not parabolic and if ϕ is an interval between the asymptotes, prove that

$$\tan^2 \phi = \frac{4(h^2 - ab)}{(a+b)^2}.$$

Ex. 21. Any two irreducible conics are similar, for which the absolute invariant $(ab - h^2)/(a+b)^2$ is the same. In particular, any two proper parabolas are similar; and so are any two proper rectangular conics.

55. A configuration of twelve points in a complex plane.

Let A_1, A_2, A_3, B_1, B_2, B_3 be six points, of which no three are in line, such that, if i, j, k is any permutation of the suffices 1, 2, 3, A_i is the pole of A_jA_k relative to the conic K_i passing through A_j, A_k, B_1, B_2, B_3.

We prove that

(α) if r, s, t also denotes any permutation of the suffices 1, 2, 3, then B_r is the pole of B_sB_t relative to the conic K_r' passing through A_1, A_2, A_3, B_s, B_t;

(β) the triangles $A_iA_jA_k$, $B_rB_sB_t$ are in perspective. The triangle $B_1B_2B_3$ thus stands in the same relation to the triangle $A_1A_2A_3$ as $A_1A_2A_3$ does to $B_1B_2B_3$.

We show further that

(γ) the six centres of perspective thus arising may be arranged as two triads C_1, C_2, C_3 and D_1, D_2, D_3 such that any two of the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, $D_1D_2D_3$ stand in the same relation to each other as do the triangles $A_1A_2A_3$, $B_1B_2B_3$ to each other; and that the six centres of perspective arising from the two triangles selected are the vertices of the remaining two triangles;

(δ) the sides of the four triangles named above form a configuration dual to that of the twelve points.

Let $A_1A_2A_3$ be taken as triangle of reference and B_1 as unit point. Then the equations of the conics K_1, K_2, K_3 are respectively

$$x^2 - yz = 0, \quad y^2 - zx = 0, \quad z^2 - xy = 0.$$

Hence B_2, B_3 have co-ordinates $(\omega^2, \omega, 1)$, $(\omega, \omega^2, 1)$ respectively, where ω is a complex cube root of unity

We may represent B_r by $(\epsilon^2, \epsilon, 1)$ where $\epsilon = 1$, ω or ω^2 according as $r = 1$, 2 or 3; and then we may represent B_s, B_t by $(\epsilon^2\eta^2, \epsilon\eta, 1)$, $(\epsilon^2\eta, \epsilon\eta^2, 1)$ respectively, where $\eta = \omega$ if r, s, t is a cyclic permutation of 1, 2, 3 and $\eta = \omega^2$ for the other three permutations.

The equation of K_r' is then

$$\epsilon^2 yz + \epsilon zx + xy = 0;$$

and the polar of B_r relative to this conic is given by

$$\epsilon x + \epsilon^2 y + z = 0,$$

which is the equation of B_sB_t

The lines A_1B_r, A_2B_s, A_3B_t are easily shown to be concurrent in the point $(\epsilon^2\eta^2, \epsilon, 1)$. Giving to ϵ, η their different possible values, we obtain co-ordinates for the six centres of perspective in the form

$$\begin{array}{lll} C_1\,(\omega^2, 1, 1), & C_2\,(1, \omega^2, 1), & C_3\,(1, 1, \omega^2), \\ D_1\,(\omega, 1, 1), & D_2\,(1, \omega, 1), & D_3\,(1, 1, \omega). \end{array}$$

The axis of perspective of the triangles $A_1A_2A_3$, $B_rB_sB_t$ has co-ordinates $(\epsilon\eta, \epsilon^2, 1)$ Again giving to ϵ, η their different possible values, the six axes of perspective are seen to be

$$\begin{array}{lll} C_2C_3\,(\omega, 1, 1), & C_3C_1\,(1, \omega, 1), & C_1C_2\,(1, 1, \omega), \\ D_2D_3\,(\omega^2, 1, 1), & D_3D_1\,(1, \omega^2, 1), & D_1D_2\,(1, 1, \omega^2). \end{array}$$

The complete proof of theorem (γ) may be achieved laboriously by elementary methods but may be dealt with more satisfactorily by studying the group of collineations under which the figure of twelve points $A_1, \ldots, D_3$ is invariant. We do not enter into full detail here but trust that the reader will be able to prove all cases of the theorem if we restrict ourselves to showing that the theorem holds when the selected triangles are, say, $C_1C_2C_3$ and $D_1D_2D_3$.

The collineation which transforms C_1, C_2, C_3, D_1 into A_1, A_2, A_3, B_1 respectively has the equations

$$x' : y' : z' = \omega(\omega x + y + z) : (x + \omega y + z) : (x + y + \omega z).$$

This collineation, of period 4, transforms the points

$$A_1, A_2, A_3, \quad B_1, B_2, B_3, \quad C_1, C_2, C_3, \quad D_1, D_2, D_3$$

into the points

$$C_1, C_3, C_2, \quad D_1, D_2, D_3, \quad A_1, A_2, A_3, \quad B_1, B_3, B_2$$

respectively. The theorem now follows from the fact that the relation between the triangles $A_1A_2A_3$, $B_1B_2B_3$ is invariant under a collineation.

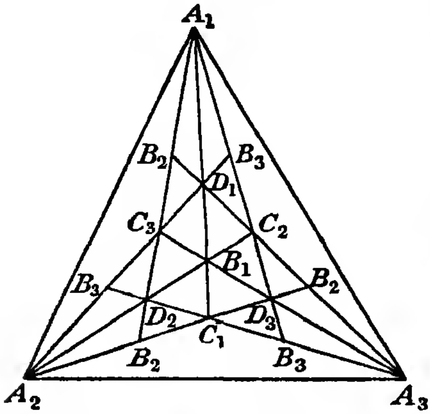


FIG. 100.—REPRESENTATION OF A CONFIGURATION OF TWELVE POINTS

Observing that the lines B_2B_3, B_3B_1, B_1B_2 have coordinates $(1, 1, 1)$, $(\omega, \omega^2, 1)$, $(\omega^2, \omega, 1)$ respectively, theorem (δ) comes immediately by application of the correlation (polarity, in fact)

$$l' : m' . n' = x : y : z.$$

On account of the complex numbers involved it is not possible to represent the incidences in the configuration by means of an ordinary real diagram. But a representation may be effected as in Fig. 100, provided that we agree to regard the three points marked B_2 as identical and likewise the three points marked B_3.

The following table, incomplete as regards the full content of theorem (γ), may be helpful in displaying the centres and axes of perspective in relation to the pairs of triangles to which they correspond.

TRIANGLES		CENTRE	AXIS
$A_1A_2A_3$	$B_1B_2B_3$	C_1	C_2C_3
$A_1A_2A_3$	$B_3B_1B_2$	C_2	C_3C_1
$A_1A_2A_3$	$B_2B_3B_1$	C_3	C_1C_2
$A_1A_2A_3$	$B_1B_3B_2$	D_1	D_2D_3
$A_1A_2A_3$	$B_2B_1B_3$	D_2	D_3D_1
$A_1A_2A_3$	$B_3B_2B_1$	D_3	D_1D_2
$C_1C_2C_3$	$D_1D_3D_2$	A_1	A_2A_3
$C_1C_2C_3$	$D_3D_2D_1$	A_2	A_3A_1
$C_1C_2C_3$	$D_2D_1D_3$	A_3	A_1A_2
$C_1C_2C_3$	$D_1D_2D_3$	B_1	B_2B_3
$C_1C_2C_3$	$D_3D_1D_2$	B_2	B_3B_1
$C_1C_2C_3$	$D_2D_3D_1$	B_3	B_1B_2

Ex. 1. The collineation, of period 3,

$$x' : y' : z' = \omega x : y : z$$

transforms the points

$$A_1, A_2, A_3, \quad B_1, B_2, B_3, \quad C_1, C_2, C_3, \quad D_1, D_2, D_3$$

into the points

$$A_1, A_2, A_3, \quad D_1, D_2, D_3, \quad B_1, B_3, B_2, \quad C_1, C_3, C_2$$

respectively.

Ex. 2. The collineation, of period 4,

$$x' : y' : z' = \omega^2(\omega^2 x + y + z) \,.\, (x + \omega^2 y + z) : (x + y + \omega^2 z)$$

transforms the points

$$A_1, A_2, A_3, \quad B_1, B_2, B_3, \quad C_1, C_2, C_3, \quad D_1, D_2, D_3$$

into the points

$$D_1, D_3, D_2, \quad C_1, C_3, C_2, \quad B_1, B_2, B_3, \quad A_1, A_2, A_3$$

respectively.

56. Statement on the position now reached. The projective plane.

It is now time to pause and look back over the whole ground which we have covered.

The outstanding feature is that we have developed a theory of the projective transformations of a line or plane (the term "collineation" being used in regard to a plane) which corresponds to the algebraic theory of homogeneous linear substitutions involving three complex variables. And, in accordance with this, the properties of configurations of points, lines and conics on which we have concentrated have been those which are invariant under projective transformations. These studies, so far as they go, constitute the *projective geometry* of the plane.

A particular aspect of the theory has been the treatment of the metrical geometry of the plane as a branch of projective geometry in which special significance is attached to two points, which we have named the circular points

The whole projective theory may evidently be applied to any aggregate of entities which are in (1, 1) correspondence, without exception, with the tersets $[x, y, z]$, excluding the terset $[0, 0, 0]$, where x, y, z belong to the field of complex numbers, if we agree to generalise the meaning of certain words in the following manner. By "point" we mean one of these entities, and by "co-ordinates of the point" we mean a corresponding triad (x, y, z). By "line" we mean the set of "points" whose co-ordinates satisfy a homo-

geneous linear equation; by "conic" the set of "points" whose co-ordinates satisfy a homogeneous quadratic equation; and so on. Such an aggregate of entities is called a *projective "plane," relative to the field of complex numbers.* And, by taking $n + 1$ complex variables instead of three, we may define in a similar way a *projective space of* n *dimensions, relative to the field of complex numbers.*

As examples of this procedure, we may mention the following cases. First, let "point" be the aggregate of all proportional triads of complex numbers, (0, 0, 0) being excluded. The aggregate of these "points" is a projective plane. Secondly, let "point" refer to a tangent to a conic, as already defined; the aggregate of these "points" is a projective line.

For a detailed discussion of the circumstances in which an aggregate of entities constitutes a projective space, the reader is referred to H. F. Baker's *Principles of Geometry* (Cambridge), where further references to the literature may be found. And for the general theory of projective space of n dimensions, there is E. Bertini's *Introduzione alla Geometria proiettiva degli Iperspazi* (Messina)

We have spoken of a "projective space, relative to the field of complex numbers." It is not necessary to consider only the algebra of complex numbers; we may consider instead the algebra of other fields of numbers Without needing to enter into all the questions involved, we illustrate these remarks by an interesting example: Fano's "plane" of seven "points."

It may be proved that there exist "finite projective planes," that is: "planes" which contain only a finite number of "points"; the number of such "points" is necessarily of the form $1 + p^n + p^{2n}$, where p is a prime number, there are $1 + p^n$ "points" on every "line" and $1 + p^n$ "lines" through every "point." Taking $p = 2$, $n = 1$, we get Fano's "plane." Taking A, B, C, D, E, F, G to denote the "points," the "lines" are given by the columns in the array

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ B & C & D & E & F & G & A \\ D & E & F & G & A & B & C. \end{array}$$

Algebraically, Fano's "plane" may be represented as follows. The "points" are all triads of integers (x, y, z), where each of x, y, z is 0 or 1, but all are not 0. Three "points" are in "line" when the sums of the corresponding co-ordinates, taken modulo 2, of two of the "points" give the co-ordinates of the third "point." For example, the "points" (1, 0, 0), (1, 1, 0), (0, 1, 0) are in "line."

The order of the group of "projective collineations" in a "plane" of $1 + p^n + p^{2n}$ "points" is $p^{3n}(p^{3n} - 1)(p^{2n} - 1)$.

Hence, by a theorem due to Cauchy, there is a "projective collineation" with period equal to any prime factor of this order; and, in any case, there is one of period $1 + p^n + p^{2n}$. If $n = 1$, every "collineation" is necessarily projective.

In the case of Fano's "plane," we may therefore look for a "collineation" of period 7. In the first representation, this corresponds to the transformation which shifts each letter cyclically one step to the right. Algebraically, it may easily be verified

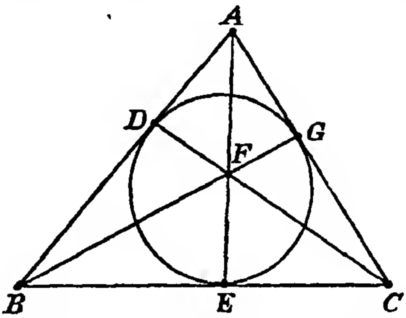


FIG. 101.—REPRESENTATION OF FANO'S "PLANE" OF SEVEN "POINTS"

that, multiplication being modulo 2, the equations $x = z'$, $y = x' + z'$, $z = y'$ represent a "collineation" of period 7.

[In regard to this example, it may be helpful to "represent" the "plane" by points of a real euclidean plane as follows. Let ABC be a triangle and let F be a point off the sides. AF, BF, CF meet BC, CA, AB respectively in E, G, D. The "lines" of the Fano "plane" are then represented by the lines already mentioned, together with the conic touching the sides of the triangle at E, G, D (Fig. 101).]

The reader who is interested in finite projective geometry should begin by turning to Veblen and Bussey, "Finite projective geometries," *Trans. Amer. Math. Soc*, 7 (1906), 241–259.

CHAPTER VII

RATIONAL CURVES

57. Geometry on a rational curve. Multiple correspondences.

(i) Regular parameterisation of a rational curve.—We suppose from now on that we are dealing with a general projective plane relative to the field of complex numbers.

Let

$$x_i = a_i(t), \quad i = 1, 2, 3,$$

be parametric equations of a rational curve C of order n. We have remarked, in section 27, and prove, in section 58, that it is always possible to choose the parameter t so that, for all but a finite number of points, there is one value of t corresponding to each point of the curve and *vice versa*. We say that C is then *regularly parameterised* and call t a *regular parameter* (or *representative parameter*, according to Baker).

(ii) Multiple points.—C being regularly parameterised, let P be a point at which t takes r distinct values $t_1, t_2, \ldots, t_r$. Then

$$a_1(t_i) : a_2(t_i) : a_3(t_i) = a_1(t_j) : a_2(t_j) : a_3(t_j).$$

The equation of a line L through P has the form

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1(t_1) & a_2(t_1) & a_3(t_1) \\ l_1 & l_2 & l_3 \end{vmatrix} = 0$$

and it meets C in the points whose parameters are given by the equation of degree n in t

$$\phi(t) \equiv \begin{vmatrix} a_1(t) & a_2(t) & a_3(t) \\ a_1(t_1) & a_2(t_1) & a_3(t_1) \\ l_1 & l_2 & l_3 \end{vmatrix} = 0.$$

This equation clearly has $t_1, t_2, \ldots, t_r$ as roots; let these occur to multiplicities $s_1, s_2, \ldots, s_r$ respectively for general values of l_1, l_2, l_3. Then P has the property that a general line through the point meets C there $s_1 + s_2 + \ldots + s_r$ times and meets C elsewhere $n - (s_1 + s_2 + \ldots + s_r)$ times. We therefore say that C has a point of *multiplicity* $s_1 + s_2 + \ldots + s_r$ at P.

(iii) Branch of a curve.—Representing the complex parameter t on an Argand diagram, the values $t_1, t_2, \ldots, t_r$ of t correspond

to r distinct points in the diagram. (Perhaps we should emphasise that "point" in the Argand diagram has a different sense from "point" in the projective plane.) Consider any domain of which the Argand point t_i is an interior point and which does not contain any other of the particular set of Argand points. The points of C which arise from the values of t in this domain are said to constitute a *branch* of C at P. For ordinary purposes, all branches of C at P arising from all domains chosen, as described, about t_i are regarded as equivalent; and we therefore speak of *the* branch of C at P which corresponds to t_i. It is evident that a general line through P meets the branch s_i times at P; therefore we say that the branch has a point of multiplicity s_i at P. The curve C has, then, r branches at P, these having multiplicities s_1, s_2, . . . , s_r at the point.

It is to be noticed that the multiplicity of P for C is greater than or equal to the number of branches of C at P.

Ex. 1. The cubic curve $y^2z = x^2(x - z)$ is parameterised regularly by the equations $x : y : z = 1 + t^2 : t + t^3 : 1$. It has two branches at the point (0, 0, 1). This point has multiplicity 1 for each branch and multiplicity 2 for the curve.

Ex. 2. The cubic curve $y^2z = x^3$ has the regular parameterisation $x : y : z = t^2 : t^3 : 1$. It has a single branch at the point (0, 0, 1) which has multiplicity 2 both for the branch and for the curve.

(iv) The idea of geometry on a curve.—The theory of the structure of multiple points on an algebraic curve, of which an indication has just been given, is developed in more detail later in this chapter. For the present we need only to bear in mind the following remarks.

The multiplicity of the point P for C comes into prominence only when we consider the curve in relation to the containing plane and may therefore be described as a notion belonging to the geometry of the plane. However, if we restrict attention to just those points which belong to the curve, we are led to the idea that, corresponding to the values t_1, t_2, . . . , t_r of t at P, C has at P a set of r *places* (to use Baker's word), one on each branch there. Relative to continuous variation of the parameter t, each place is distinct; but, relative to the plane, each place coincides with the same point in the plane. The idea of place, then, belongs to the *geometry on the curve.*

(v) Multiple correspondences.—Let λ, μ be variables connected by a relation of the form $A(\lambda, \mu) = 0$, where

$$A(\lambda, \mu) = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij}\lambda^i\mu^j;$$

which may be rewritten in the form

$$A(\lambda, \mu) \equiv a_0\mu^n + a_1\mu^{n-1} + \ldots + a_n,$$

where the a_i are polynomials of order m in λ; or in the form

$$A(\lambda, \mu) \equiv b_0\lambda^m + b_1\lambda^{m-1} + \ldots + b_m,$$

where the b_i are polynomials of order n in μ.

Each value of λ determines n values of μ, in general distinct and variable with λ; and each value of μ determines m values of λ, in general distinct and variable with μ. If, then, λ and μ are the non-homogeneous parameters of places P and Q respectively on a regularly parameterised rational curve C, there is set up on C a correspondence between the places. Each place P gives rise to n corresponding places Q, in general distinct and variable with P; and each place Q inversely gives rise to m places P, in general distinct and variable with Q The correspondence is called an *algebraic multiple*, more particularly an (m, n), *correspondence*, and m, n are called its *indices*.

If $B_1, B_2, \ldots, B_n$ are the positions on the curve of Q which arise from the position A of P, we say that A *corresponds directly* to $B_1, B_2, \ldots, B_n$, and symbolise this by writing

$$T(A) = B_1 + B_2 + \ldots + B_n.$$

If we wish only to indicate that B_i is a particular place arising from A, we symbolise this by writing $T(A) \supset B_i$.

For a particular value of λ, the resulting equation in μ has repeated roots. If such a root occurs r times, the place B_j, whose parameter has this value, is to be counted r times in the set of places to which A corresponds directly We write $T(A) \supset rB_j$.

Similarly, if $A_1, A_2, \ldots, A_m$ are the positions of P on the curve which arise from the position B of Q, we say that B *corresponds inversely* to $A_1, A_2, \ldots, A_m$, and symbolise this by writing

$$T^{-1}(B) = A_1 + A_2 + \ldots + A_m,$$
$$T^{-1}(B) \supset A_i;$$

and, as just indicated, it may be necessary, for particular positions of B, to count a corresponding place A_i more than once.

It is clear that $T(A_i) \supset B$, and $T^{-1}(B_i) \supset A$.

The correspondence is said to be *reducible* when the polynomial $A(\lambda, \mu)$ factorises; otherwise it is said to be *irreducible*.

The particular correspondence, represented by the equation $\lambda = \mu$, is called the *identical correspondence*. For this the letter I is used in place of T. It is clear that I^{-1} is the same as I.

It is useful to remark that a non-singular bilinear change of parameter on the curve gives a new regular parameter and does not alter the indices of the correspondence.

Thus if

$$\lambda = (p\lambda' + q)/(r\lambda' + s), \quad \mu = (p\mu' + q)/(r\mu' + s),$$

with $ps \neq qr$, so that to each place corresponds one value of the new parameter, and *vice versa*, the equation $A(\lambda, \mu) = 0$ becomes

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij}(p\lambda' + q)^i(r\lambda' + s)^{m-i}(p\mu' + q)^j(r\mu' + s)^{n-j} = 0,$$

which may be rearranged in the form

$$A'(\lambda', \mu') \equiv \sum_{k=0}^{m} \sum_{l=0}^{n} \alpha_{kl}'\lambda'^k\mu'^l = 0;$$

and to each factor of $A(\lambda, \mu)$ corresponds a factor of $A'(\lambda', \mu')$, and *vice versa.*

Ex. 3. An irreducible (1, 1) correspondence is a projectivity.

Ex. 4. An (m, n) correspondence is determined by $mn + m + n$ pairs of corresponding places P, Q

Ex. 5. If $T(P)$ contains a fixed place, with parameter b, the correspondence is reducible, and $\mu - b$ divides $A(\lambda, \mu)$. And if $T^{-1}(Q)$ contains a fixed place, with parameter a, the correspondence is reducible, and $\lambda - a$ divides $A(\lambda, \mu)$.

(vi) United places of a correspondence. Chasles' formula.— If $T(P) \supset P$, then also $T^{-1}(P) \supset P$. We say that P is a *united place* of the correspondence. The united places are determined by the equation

$$A(\lambda, \lambda) = 0,$$

which is of order $m + n$ in λ. Therefore the number u of united places is given, in general, by

$$u = m + n.$$

This is *Chasles' formula.*

Exceptionally it may happen that the equation $A(\lambda, \lambda) = 0$ has repeated roots In order to preserve Chasles' formula, we decide that, if a root (which may be ∞) occurs to multiplicity r, the place whose parameter has this value is to count r times in the set of united places.

Or it may happen that the equation $A(\lambda, \lambda) = 0$ is satisfied identically by all values of λ. This is a necessary and sufficient condition for $\lambda - \mu$ to divide $A(\lambda, \mu)$. The correspondence is then reducible, unless it is I itself, and every place is a united place.

In order to investigate the behaviour of the correspondence in regard to a particular place on the curve it is sometimes convenient to rely on the result at the end of part (v) and, if necessary, effect a preliminary linear change of parameter so that at the

particular place the new parameter takes an assigned value, say 0. With this in mind, it is easy to prove the following statements.

Ex. 7. If $T(P) \supset rP$, $r \geqslant 1$, then P counts, in general, just once in the set of united places.

Ex. 8. If $T(P) \supset rP$, and $T^{-1}(P) \supset sP$, with $r \geqslant 2$, $s \geqslant 2$, then P counts, in general, just twice in the set of united places.

Ex. 9. If $T(P) \supset rB$, where B is fixed, then B counts r times in the set of united places.

(vii) Branch places of a correspondence. Zeuthen's formula.—Although, in general, the places in the set $T(P)$ are distinct, there are positions of P for which two or more places of the set $T(P)$ coincide. Such a position of P is called a *branch place* of the transformation T. If only two places in the set $T(P)$ coincide, the branch place P is said to be *simple*; and the place which is counted twice in $T(P)$ is called a *double place* of T.

The parameters of the branch places of T are obtained by eliminating μ from the equations

$$a_0\mu^n + a_1\mu^{n-1} + \ldots + a_n = 0,$$
$$na_0\mu^{n-1} + (n-1)a_1\mu^{n-2} + \ldots + a_{n-1} = 0.$$

The resultant is of order $2(n-1)$ in the coefficients a_i, and therefore of order $2m(n-1)$ in λ. If, as occurs in general, the roots of this equation in λ are distinct, the number ν' of branch places of T is therefore given by

$$\nu' = 2m(n-1).$$

Similarly, there is a set of branch places associated with the inverse transformation T^{-1}; and the number ν of these is given, in general, by

$$\nu = 2n(m-1).$$

Combining the two equations just obtained, we have

$$\nu - \nu' = 2(m-n),$$

a formula due to *Zeuthen*.

Zeuthen's and Chasles' formulae are capable of generalisation with specific modifications to the theory of multiple correspondences between irrational curves, and are of fundamental importance in that theory.

(viii) Symmetrical correspondences.—A correspondence is said to be *symmetrical* if $m = n$ and $\alpha_{ij} = \alpha_{ji}$ for all i, j. Such a correspondence is determined by $\frac{1}{2}m(m+3)$ pairs of corresponding places. It has the property that, if $T(P) \supset Q$, then $T(Q) \supset P$.

Conversely we prove that, if an irreducible correspondence has

the properties that $m = n$ and that, for all positions of P, $T(P) \supset Q$ implies $T(Q) \supset P$, then the correspondence is symmetrical or else is the identical correspondence.

The identical correspondence clearly satisfies the conditions of the hypothesis. Apart from this case, the hypothesis of irreducibility implies that at least one place on the curve is not a united place. Choosing such a place, we may suppose that a non-singular bilinear change of parameter has been effected, if necessary, so that the parameter of the place has the value ∞; then $\alpha_{mm} \neq 0$.

When λ takes the value c, let the corresponding values of μ be $d_1, d_2, \ldots, d_m$. Then, when μ takes the value c, the corresponding values of λ will be $d_1, d_2, \ldots, d_m$ The equation in μ

$$a_0(c)\mu^m + a_1(c)\mu^{m-1} + \ldots + a_m(c) = 0$$

has therefore the same roots as the equation in λ

$$b_0(c)\lambda^m + b_1(c)\lambda^{m-1} + \ldots + b_m(c) = 0.$$

Therefore

$$\frac{a_0(c)}{b_0(c)} = \frac{a_1(c)}{b_1(c)} = \ldots = \frac{a_m(c)}{b_m(c)};$$

and these equalities hold for all values of c.

Expressing the a_i, b_i in terms of the α_{ij}, it follows that

$$\frac{\sum_i \alpha_{ij} c^i}{\sum_i \alpha_{ji} c^i} = \frac{\sum_k \alpha_{km} c^k}{\sum_k \alpha_{mk} c^k}, \text{ for all } c, j;$$

that is

$$\sum_{i,k} (\alpha_{ij}\alpha_{mk} - \alpha_{ji}\alpha_{km})c^{i+k} = 0, \text{ for all } c, j;$$

and therefore

$$\sum_{i+k=\text{constant}} (\alpha_{ij}\alpha_{mk} - \alpha_{ji}\alpha_{km}) = 0, \text{ for all } j.$$

In particular, by taking $i + k = 2m$, we have

$$\alpha_{mj}\alpha_{mm} - \alpha_{jm}\alpha_{mm} = 0, \text{ for all } j;$$

whence, since $\alpha_{mm} \neq 0$,

$$\alpha_{mj} = \alpha_{jm}, \text{ for all } j.$$

The rest follows by the principle of finite induction. Let us suppose that we have proved that, for all j, $\alpha_{ij} = \alpha_{ji}$ for $i > r$; we have to prove that $\alpha_{rj} = \alpha_{jr}$ for all j. Now

$$\sum_{i+k=r+m} (\alpha_{ij}\alpha_{mk} - \alpha_{ji}\alpha_{km}) = 0;$$

therefore

$$(\alpha_{rj}\alpha_{mm} - \alpha_{jr}\alpha_{mm}) = - \sum_{i+k=r+m}' (\alpha_{ij}\alpha_{mk} - \alpha_{ji}\alpha_{km}),$$

where the dash indicates that i, k are not to take the values r, m respectively. Since $\alpha_{mk} = \alpha_{km}$, as already proved, the expression on the right may be rewritten as

$$-\sum_{i+k=r+m}{}' \alpha_{mk}(\alpha_{ij} - \alpha_{ji}).$$

In this summation the pairs of values available for i, k are $r+1$, $m-1$; $r+2$, $m-2$; ...; m, $m-r$; and, for each of these available values of i, we have $\alpha_{ij} = \alpha_{ji}$. Therefore the sum Σ' is zero, and $\alpha_{rj} = \alpha_{jr}$. Thus, for all values of i, j, $\alpha_{ij} = \alpha_{ji}$.

Ex. 10. In a symmetrical correspondence, if $T(P) \supset rP$, $r \geqslant 2$, then P counts, in general, twice in the set of united places. Conversely, if P counts twice in the set of united places, $T(P) \supset rP$, with $r \geqslant 2$.

Ex. 11. If a symmetrical (m, m) correspondence has more than $2m$ united places, each united place being counted with its appropriate multiplicity, the correspondence is reducible and its equation may be expressed in the form $(\lambda - \mu)^2 B(\lambda, \mu) = 0$, where $B(\lambda, \mu) = 0$ is the equation of a symmetrical $(m-2, m-2)$ correspondence, or a constant in the case $m = 2$.

[We may usefully remark on the proof of this statement. Let the equation of the correspondence be $\Sigma\alpha_{ij}\lambda^i\mu^j = 0$, and put $\lambda + \mu = 2x$, $\lambda - \mu = 2y$. The equation then becomes $f(x, y) = 0$, where

$$f(x, y) \equiv \Sigma\alpha_{ij}(x + y)^i(x - y)^j.$$

Since $\alpha_{ij} = \alpha_{ji}$, $f(x, y) \equiv f(x, -y)$. The polynomial $f(x, y)$ therefore contains only even powers of y. And since there are more than $2m$ united places (which are given by $y = 0$), we have, identically, $f(x, 0) = 0$. Therefore y^2 divides the polynomial $f(x, y)$. The remainder of the proof follows from the preceding text.]

(ix) Involutions.

(*a*) Let A be any place on the curve, and let $T(A) = B_1 + B_2 + \ldots + B_m$, the correspondence being supposed irreducible. Then it may happen, as in the case of an ordinary involution of pairs of places, that, for all positions of A and for all values of i, $T(B_i) = A + B_1 + \ldots + B_{i-1} + B_{i+1} + \ldots + B_m$. The correspondence, which is then necessarily symmetrical, is said to be *involutory*, and the aggregate of the sets of places $A, B_1, B_2, \ldots, B_m$ is called an *involution of order* $m + 1$; each place on the curve belongs to just one such set.

We prove that the sets of the involution are determined by an equation of the form

$$(p_0\tau^{m+1} + p_1\tau^m + \ldots + p_{m+1}) + \sigma(q_0\tau^{m+1} + q_1\tau^m + \ldots + q_{m+1}) = 0,$$

there being one value of σ for each set.

Let the equation of the correspondence be given in the form

$$a_0(\lambda)\mu^m + a_1(\lambda)\mu^{m-1} + \ldots + a_m(\lambda) = 0,$$

where the a_i are polynomials of order m in λ, with no common factor, and at least one of them has a non-zero term in λ^m. Then let $m + 1$ polynomials $c_i(\lambda)$ be defined by the identity in μ

$$\begin{aligned} &a_0(\lambda)\mu^{m+1} + c_1(\lambda)\mu^m + \ldots + c_{m+1}(\lambda) \\ &\equiv (\mu - \lambda)(a_0(\lambda)\mu^m + a_1(\lambda)\mu^{m-1} + \ldots + a_m(\lambda)). \end{aligned}$$

The polynomials $c_i(\lambda)$ are connected with the $a_i(\lambda)$ by the relations

$$\begin{aligned} c_i(\lambda) &\equiv a_i(\lambda) - \lambda a_{i-1}(\lambda), \quad i = 1, 2, \ldots, m, \\ c_{m+1}(\lambda) &\equiv -\lambda a_m(\lambda). \end{aligned}$$

From these it follows that at least one polynomial $c_i(\lambda)$ has a non-zero term in λ^{m+1}.

Next, let $\theta_1, \theta_2, \ldots, \theta_m$ be the values of μ corresponding to the value θ of λ. Since the correspondence is irreducible, θ differs from each of $\theta_1, \theta_2, \ldots, \theta_m$, except for particular values of θ. Moreover, two of the numbers $\theta_1, \theta_2, \ldots, \theta_m$ are equal if and only if θ is the parameter of one of the finite number of branch places of the correspondence. Thus, except for a finite number of sets, the numbers $\theta, \theta_1, \theta_2, \ldots, \theta_m$ are distinct.

We now observe that

$$\begin{aligned} &a_0(\theta)\mu^{m+1} + c_1(\theta)\mu^m + \ldots + c_{m+1}(\theta) \\ &\qquad \equiv a_0(\theta) \,.\, (\mu - \theta)(\mu - \theta_1) \ldots (\mu - \theta_m), \end{aligned}$$

and, by the involutory hypothesis, that

$$\begin{aligned} &a_0(\theta_1)\mu^{m+1} + c_1(\theta_1)\mu^m + \ldots + c_{m+1}(\theta_1) \\ &\qquad \equiv a_0(\theta_1) \,.\, (\mu - \theta)(\mu - \theta_1) \ldots (\mu - \theta_m). \end{aligned}$$

Therefore

$$\frac{c_i(\theta)}{c_i(\theta_1)} = \frac{a_0(\theta)}{a_0(\theta_1)}, \; i = 1, 2, \ldots, m + 1.$$

Replacing θ_1 by $\theta_2, \ldots, \theta_m$ in turn, we have

$$\frac{c_i(\theta)}{a_0(\theta)} = \frac{c_i(\theta_1)}{a_0(\theta_1)} = \ldots = \frac{c_i(\theta_m)}{a_0(\theta_m)}, \quad i = 1, 2, \ldots, m + 1.$$

Let $c_r(\tau)$ have a non-zero term in τ^{m+1}, and let $\sigma_r = -c_r(\theta)/a_0(\theta)$ Then the polynomial in τ, of order $m + 1$,

$$c_r(\tau) + \sigma_r a_0(\tau)$$

vanishes for the $m + 1$ distinct values $\theta, \theta_1, \ldots, \theta_m$ of τ. That is to say, the sets of the involution are determined by the equation

$$c_r(\tau) + \sigma a_0(\tau) = 0,$$

each set corresponding to a particular value of σ.

It is to be noted that, if $c_i(\tau)$ has a zero term in τ^{m+1}, then $c_i(\tau) + \sigma_i a_0(\tau)$, where $\sigma_i = -c_i(\theta)/a_0(\theta)$, is identically zero, being a polynomial with more zeros than its order. Further, if $c_s(\tau)$ is another polynomial with a non-zero term in τ^{m+1}, there is an identity of the form $c_s(\tau) + \sigma_s a_0(\tau) \equiv k_{sr}(c_r(\tau) + \sigma_r a_0(\tau))$, since each side is proportional to $(\tau - \theta)(\tau - \theta_1) \ldots (\tau - \theta_m)$.

Ex. 12. Show that

$$a_0(\theta)\mu^{m+1} + c_1(\theta)\mu^m + \ldots + c_{m+1}(\theta) \equiv a_0(\theta)A(\mu) + c_r(\theta)C(\mu),$$

where

$$A(\mu) \equiv \mu^{m+1} + \sum_{i=1}^{m+1}(k_{ir}\sigma_r - \sigma_i)\mu^{m+1-i}, \quad C(\mu) \equiv \sum_{i=1}^{m+1} k_{ir}\mu^{m+1-i}$$

(*b*) Conversely, let α, β be any two different values of τ which satisfy, for some value of σ, the equation

$$P(\tau) + \sigma Q(\tau) = 0,$$

where the polynomials

$$\begin{aligned} P(\tau) &\equiv p_0\tau^{m+1} + p_1\tau^m + \ldots + p_{m+1}, \\ Q(\tau) &\equiv q_0\tau^{m+1} + q_1\tau^m + \ldots + q_{m+1} \end{aligned}$$

have no common factor.

Then, since

$$P(\alpha) + \sigma Q(\alpha) = 0 = P(\beta) + \sigma Q(\beta),$$

we have

$$P(\alpha)Q(\beta) - P(\beta)Q(\alpha) = 0.$$

Thus the equation

$$A(\lambda, \mu) \equiv \frac{P(\lambda)Q(\mu) - P(\mu)Q(\lambda)}{\lambda - \mu} = 0$$

is satisfied by $\lambda = \alpha$, $\mu = \beta$.

This equation represents a symmetrical (m, m) correspondence between λ, μ Regarding λ as any one root of the equation

$$P(\tau) + \sigma Q(\tau) = 0,$$

for the value $-P(\lambda)/Q(\lambda)$ of σ, the remaining roots of this equation in τ are the values of μ given by $A(\lambda, \mu) = 0$. The (m, m) correspondence is therefore involutory.

(x) Extensions of the theory.—Just as in the case of projective transformations, it is clear that the algebra of this section may be used to develop, in an obvious way, a theory of multiple correspondences between the places of two rational curves, between the lines (using the term in the same sense as "place")

of two rational envelopes, and between the places of a rational curve and the lines of a rational envelope. In the case of a multiple correspondence between the elements of two different rational aggregates, there is no place for the notion of *united element*; but the notion of *branch element* continues to apply, with fundamental importance.

If, then, we regard σ as a regular parameter on a rational curve C', the theorem of part (ix) asserts that there is an $(m + 1, 1)$ correspondence between the places on the curve C, containing the involution, and the places on C', such that the sets of the involution correspond to the points of C'. We say that the places on C' *map* or *represent* the sets of the involution; and, since C' is rational, we say that the involution is *rational.*

The theorem of part (ix) may thus be expressed in the terms: every involution on a rational curve is rational. This is a theorem of considerable importance in algebraic geometry.

58. Lüroth's theorem.

Let C be a curve, defined by the equations

$$x_1 : x_2 : x_3 = f_1(\theta) : f_2(\theta) : f_3(\theta),$$

where the f_i are polynomials which have no common factor involving θ and are not all constants We prove that there exists, and show how to find, a regular parameter, expressible as a rational function of θ.

First, a necessary and sufficient condition that C should be a line is that constants a_1, a_2, a_3, not all zero, exist so that

$$a_1 f_1(\theta) + a_2 f_2(\theta) + a_3 f_3(\theta) \equiv 0.$$

In this case suppose that $a_3 \neq 0$; $f_1(\theta)$ and $f_2(\theta)$ cannot both be zero, so suppose $f_2(\theta) \neq 0$ and define σ by the equation $f_1(\theta) = \sigma f_2(\theta)$ if $f_1(\theta) \neq 0$, and by $f_3(\theta) = \sigma f_2(\theta)$ if $f_1(\theta) \equiv 0$.

The equations of C then take either the form

$$x_1 : x_2 : x_3 = a_3\sigma : a_3 : -a_1\sigma - a_2,$$

or else $$x_1 : x_2 : x_3 = 0 : 1 : \sigma.$$

Thus σ is a regular parameter.

Now let us suppose that C is not a line. Then the f_i are linearly independent; in particular, no two are proportional.

Let (ξ_i) be a general point on the curve. The equations

$$\xi_i f_j(\theta) - \xi_j f_i(\theta) = 0, \quad i, j = 1, 2, 3, \quad i \neq j,$$

have a certain number m of distinct roots $\theta_1, \theta_2, \ldots, \theta_m$ in common. If $m = 1$, θ is a regular parameter, and *vice versa*; we suppose therefore that $m > 1$ and then we have

$$f_1(\theta_i) : f_2(\theta_i) : f_3(\theta_i) = f_1(\theta_j) : f_2(\theta_j) : f_3(\theta_j), \quad i, j = 1, \ldots, m.$$

None of the numbers θ_i is fixed; for then the $f_i(\theta)$ would be proportional to constants, which is excluded. Nor is any θ_i a multiple root for all the equations; for then the equations

$$\xi_i f_j(\theta) - \xi_j f_i(\theta) = 0, \quad \xi_i f_j'(\theta) - \xi_j f_i'(\theta) = 0$$

would have a common root, and (ξ_i), instead of being a general point on C, would be at one of the intersections of C with the curve

$$x_1 : x_2 : x_3 = f_1'(\theta) : f_2'(\theta) : f_3'(\theta).$$

Replacing ξ_i by $f_i(\theta_r)$, it follows from these remarks that $M(\theta, \theta_r)$, the greatest common divisor of the polynomials.

$$f_i(\theta_r)f_j(\theta) - f_j(\theta_r)f_i(\theta), \quad i, j = 1, 2, 3, i \neq j,$$

is given by

$$M(\theta, \theta_r) \equiv k(\theta - \theta_1)(\theta - \theta_2) \dots (\theta - \theta_m),$$

where k is independent of θ.

Now let us take any line L and denote by λ, μ values of a regular parameter on this line. Then the equation

$$A(\lambda, \mu) \equiv \frac{M(\lambda, \mu)}{\mu - \lambda} = 0$$

has coefficients which are expressible rationally in terms of the coefficients of f_1, f_2, f_3 and determines a symmetrical $(m - 1, m - 1)$ correspondence on L. Regarding μ as any one of the set of numbers $\theta_1, \theta_2, \dots, \theta_m$ arising from the point $(\xi_i) = (f_i(\mu))$, the corresponding values of λ given by $A(\lambda, \mu) = 0$ are the remainder of this set of numbers. The $(m - 1, m - 1)$ correspondence is therefore involutory.

Writing $M(\lambda, \mu)$ $(= (\mu - \lambda)A(\lambda, \mu))$ in the form

$$a_0(\lambda)\mu^m + c_1(\lambda)\mu^{m-1} + \dots + c_m(\lambda),$$

we have seen (section 57 (ix)) that, if $c_s(\lambda)$ has a non-zero term in λ^m (and this occurs for some value of s), then $\theta_1, \theta_2, \dots, \theta_m$ are the roots of the equation

$$c_s(\theta) + \sigma a_0(\theta) = 0,$$

for a definite value of σ. There is thus one value of σ for every point (ξ_i), and *vice versa*.

The polynomial in θ

$$\xi_i f_j(\theta) - \xi_j f_i(\theta)$$

vanishes for $\theta = \theta_1, \theta_2, \dots, \theta_m$ and is therefore divisible by $c_s(\theta) + \sigma a_0(\theta)$. Expressing the fact that the remainder, which is linear in ξ_i, ξ_j and rational in σ, is identically zero, we obtain the ratio $\xi_i : \xi_j$ as the ratio of two polynomials in σ. Performing this

operation for all values of i, j, we derive equations for C with σ as a regular parameter.

Ex. 1. Discuss the parameterisation of the curve

$$x_1 : x_2 : x_3 = \theta(\theta^4 + 1) : (\theta^2 - 1)(\theta^4 + 1) : \theta^3.$$

[Here $M(\lambda, \mu) \equiv \mu^2\lambda + \mu(1 - \lambda^2) - \lambda$. We therefore form the expression $(1 - \theta^2) + \sigma\theta$, and divide it into each of the expressions

$$(x_1(\theta^2 - 1) - x_2\theta)(\theta^4 + 1),$$
$$x_1\theta^3 - x_3\theta(\theta^4 + 1);$$

but clearly the process will give the same ultimate result if instead we divide into

$$x_1(\theta^2 - 1) - x_2\theta,$$
$$x_1\theta^2 - x_3(\theta^4 + 1).$$

The remainders are respectively

$$\theta(\sigma x_1 - x_2),$$
$$(\sigma\theta + 1)(x_1 - (2 + \sigma^2)x_3).$$

Since these vanish identically, we have

$$x_1 : x_2 : x_3 = (2 + \sigma^2) : \sigma(2 + \sigma^2) : 1$$

as the required regular parametric equations, with $\sigma = (\theta^2 - 1)/\theta$.]

Ex. 2. Show that the ordinary equation of this curve is $x_2{}^2x_3 = x_1{}^2(x_1 - 2x_3)$, that the curve has a double point at $(0, 0, 1)$, there being two branches there.

59. Algebraic (2, 2) correspondences.

In the previous two sections we have been concerned with geometry on any rational curve. It is worth while now to illustrate the theory by reference to (2, 2) correspondences; and, we shall see, there are interesting aspects of geometry in the plane associated with the particular rational curve selected for discussion.

(i) Symmetric (2, 2) correspondence on a conic.—We choose for consideration an irreducible conic C, which we may take to have the regular parametric equations $x : y : z = t^2 : t : 1$; here "point" and "place" have the same significance. On C we consider the general symmetric (2, 2) correspondence; the equation of the correspondence may be put in the form

$$\lambda^2(a\mu^2 + h\mu + g) + \lambda(h\mu^2 + b\mu + f) + (g\mu^2 + f\mu + c) = 0,$$

which may also be written as

$$a\lambda^2\mu^2 + b\lambda\mu + c + f(\lambda + \mu) + g((\lambda + \mu)^2 - 2\lambda\mu) + h\lambda\mu(\lambda + \mu) = 0.$$

In the first form the equation shows that a necessary and sufficient condition for two points on C to correspond under T is that they should be conjugate relative to the conic S whose equation is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0;$$

and, in the second form, the equation shows that another necessary and sufficient condition is that the line joining the two points (a tangent to C when the points coincide) should belong to the conic-envelope Σ given by

$$an^2 + bnl + cl^2 - flm + g(m^2 - 2nl) - hmn = 0.$$

The united points of T are the points common to C, S; these are also the points of contact of the tangents to C which belong to Σ. The harmonic conic-envelope of C, S is Σ.

Ex. 1. A necessary and sufficient condition that the points corresponding directly to any point P on C should be the points corresponding directly and inversely to P under a projectivity is that the conic associated with Σ should have double contact with C.

[It is enough to remark that when this double contact occurs the equation of Σ takes the form

$$(pl + qm + rn)^2 = k(m^2 - 4nl),$$

and the equation of T is then

$$(p - q(\lambda + \mu) + r\lambda\mu)^2 = k(\lambda - \mu)^2.$$

This equation may be written as

$$\{p - q(\lambda + \mu) + r\lambda\mu - k^{\frac{1}{2}}(\lambda - \mu)\}\{p - q(\lambda + \mu) + r\lambda\mu + k^{\frac{1}{2}}(\lambda - \mu)\} = 0.$$

Each factor on the left, equated to zero, represents a projectivity; and each projectivity is the inverse of the other.]

Ex. 2. A necessary and sufficient condition that the points which correspond under T to P should be the mates of P in a pair of involutory projectivities is that Σ should consist of two pencils of lines.

(ii) A theorem due to Poncelet.—Let C be an irreducible conic and Σ be an irreducible conic-envelope such that there is a polygon $A_1A_2 \dots A_n$ $(n \geqslant 3)$ whose vertices $A_1, A_2, \dots, A_n$ lie on C and whose sides $A_1A_2, A_2A_3, \dots, A_nA_1$ belong to Σ. We prove that there is an infinity of such polygons.

This theorem has already been considered from another point of view in the cases $n = 3, 4$ in sections 22, 43 respectively.

Let P_1 be a general point on C. Through P_1 pass two lines of Σ; we choose one and let this meet C again at P_2; through P_2 passes one more line of Σ which meets C again at a point P_3; . . .; through P_n passes one more line of Σ which meets C again at a point P_{n+1} (Fig. 102). We have to prove that P_{n+1} is P_1.

Suppose that this is not the case. Then, in virtue of the choice available at the outset in this construction, P_1, P_{n+1} are corresponding points in a symmetric (2, 2) correspondence, which is clearly algebraic. Now if we take P_1 at any one of the points $A_1, A_2, \ldots, A_n$ both of the corresponding positions of P_{n+1} coincide with P_1. Hence, by the result of *Ex.* 10 of section 57 (viii), each of the points $A_1, A_2, \ldots, A_n$ counts twice in the set of united places of the correspondence. The correspondence thus has $2n$ ($\geqslant 6$) united places, it is therefore the identity correspondence taken twice by *Ex.* 11 of section 57 (viii). This contradicts the assumption which is therefore false.

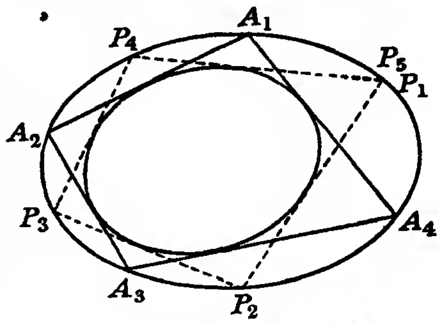


FIG. 102.—PONCELET'S THEOREM IN THE CASE $n = 4$

Hence P_{n+1} is P_1; and the theorem follows.

Ex. 3 Referring to the text, show that, in general, if there is no polygon such as $A_1A_2 \ldots A_n$, then P_1P_n belongs to a conic-envelope Σ', and C belongs to the pencil of conics determined by the conics associated with Σ, Σ'.

(iii) The involutory case.—We now enquire into the case where T is involutory. Each point on C is then one of a triad of points which are the vertices of a triangle self-polar relative to S and whose sides belong to Σ. Therefore C is outpolar to S and Σ is inpolar to S. A necessary and sufficient condition for this relationship is the vanishing of a relative invariant of C, S, which is here expressed by

$$ca - g^2 = hf - bg.$$

This condition, together with other information, may also be reached by direct algebra as follows. In the notation of section 57, we form the expression $(\mu - \lambda)A(\lambda, \mu)$, which here is

$$\mu^3(a\lambda^2 + h\lambda + g) + \mu^2((h\lambda^2 + b\lambda + f) - \lambda(a\lambda^2 + h\lambda + g)) + \\ \mu((g\lambda^2 + f\lambda + c) - \lambda(h\lambda^2 + b\lambda + f)) + (-\lambda(g\lambda^2 + f\lambda + c));$$

this is the polynomial $\mu^3a_0(\lambda) + \mu^2c_1(\lambda) + \mu c_2(\lambda) + c_3(\lambda)$ in the

general notation. We have next to express the fact that, given any number σ, there exist numbers σ', σ'', k', k'' such that

$$c_1(\lambda) - \sigma' a_0(\lambda) \equiv k'(c_3(\lambda) - \sigma a_0(\lambda)),$$
$$c_2(\lambda) - \sigma'' a_0(\lambda) \equiv k''(c_3(\lambda) - \sigma a_0(\lambda)).$$

The arithmetic is straightforward and leads to the necessary and sufficient condition $ca - g^2 = hf - bg$.

The sets of the involution are then determined by any one of the equations $c_i(\lambda) - \theta a_0(\lambda) = 0$, in which θ varies from one set to another. For example, the triads are determined by the equation

$$-\lambda(g\lambda^2 + f\lambda + c) - \theta(a\lambda^2 + h\lambda + g) = 0;$$

therefore each set consists of the points common to C and a conic of the pencil

$$(gxy + fzx + cyz) + \theta(azx + hyz + gz^2) = 0,$$

excluding the fixed common point (1, 0, 0).

Ex. 4. P, Q, R and P', Q', R' are two triads of the involution; A is any fixed point on C. An arbitrary conic through P, Q, R, A meets an arbitrary conic through P', Q', R', A in three points B, C, D besides A. Prove that the sets of the involution are the sets cut on C, apart from A, by the conics of the pencil through A, B, C, D.

60. Remarks on algebraic curves in general, with particular references to rational curves.

In this chapter we are primarily concerned with rational curves. It is desirable at this stage to make some elementary observations which apply to all algebraic curves in a plane and, on the basis of these observations, to gain further insight into the structure of rational curves.

(1) Linear conditions.—We consider first the linear system of all curves of order n. The number of homogeneous products of degree n in three variables is $\frac{1}{2}(n+1)(n+2)$; in the equation of a curve only the ratios of the coefficients of these terms are significant; the system is therefore $\infty^{\frac{1}{2}n(n+3)}$, or of freedom $\frac{1}{2}n(n+3)$.

To constrain a curve to pass through a given point is one linear condition on the coefficients in the equation of the curve. There is therefore just one curve of order n which can be put through $\frac{1}{2}n(n+3)$ given points, provided that these present independent conditions: that this is the case if the points are in sufficiently general positions we now prove. Let us suppose

that it has been proved that r points can be found to present independent linear conditions, with $r < \frac{1}{2}n(n+3)$; we have just seen that $r \geqslant 1$. Taking r such points, the curves of order n through these form a linear system of freedom $\frac{1}{2}n(n+3) - r$. These curves may, as examples to follow will show, contain other fixed points in consequence of passing through the r assigned points. An $(r+1)$th point, different from any of the fixed common points, will then impose one more independent linear condition. Proceeding inductively, we may thus select, in many ways, $\frac{1}{2}n(n+3)$ points of sufficient generality to present independent linear conditions and so to belong to just one curve of order n.

Ex. 1. The pencil of cubic curves $u_1u_2u_3 + \lambda v_1v_2v_3 = 0$, where the $u_i = 0$, $v_i = 0$ are lines, no three of which are concurrent, contain the nine points given by $u_i = v_j = 0$, $i, j = 1, 2, 3$. The nine points therefore present eight, or fewer, independent linear conditions to cubic curves. Prove that, in fact, they present precisely eight independent conditions.

(ii) **Intersections of a curve and a line.**—Let (y_i), (z_i) be two different points in the plane of the curve C, determined by the vanishing of a homogeneous polynomial $f(x_1, x_2, x_3)$ of degree n. Every point on the line joining the two points has co-ordinates of the form $(\lambda y_i + \mu z_i)$. The ratios of λ, μ for the points on the line which belong also to C are given by the equation

$$f(\lambda y_1 + \mu z_1, \quad \lambda y_2 + \mu z_2, \quad \lambda y_3 + \mu z_3) = 0.$$

This equation may be expressed as

$$\lambda^n f(y_1, y_2, y_3) + \frac{\lambda^{n-1}\mu}{\underline{|1}} \nabla f(y_1, y_2, y_3)$$
$$+ \frac{\lambda^{n-2}\mu^2}{\underline{|2}} \nabla^2 f(y_1, y_2, y_3) + \ldots = 0,$$

where

$$\nabla \equiv z_1 \frac{\partial}{\partial y_1} + z_2 \frac{\partial}{\partial y_2} + z_3 \frac{\partial}{\partial y_3}.$$

If this equation is satisfied identically, the curve is reducible and consists of the line together with a curve of order $n - 1$. Otherwise, it follows that the curve meets the line in precisely n points, some of which may coincide or be infinitely near, according to the conventional language adopted; this we have remarked on in an earlier section.

More generally, a similar argument proves that a curve of order n meets a rational curve of order m in nm places, some of which may be infinitely near on the rational curve, or else contains the curve as one component, for which necessary and sufficient

conditions are $m < n$ and that there should be at least $mn + 1$ places of intersection. It follows that $mn + r$ $(r > 1)$ places on the rational curve present just $mn + 1$ independent linear conditions to curves of order n.

Ex. 2. A set of $n + r$ $(r > 1)$ points on a line presents just $n + 1$ independent linear conditions to curves of order n.

(iii) Simple and multiple points.—If $f(y_1, y_2, y_3) = 0$ but $\nabla f(y_1, y_2, y_3)$ is not identically zero, the point (y_i) is said to be *simple* on C. Then every line through the point has $n - 1$ intersections with C, different from (y_i) except in the case of the line

$$x_1 \frac{\partial f(y_1, y_2, y_3)}{\partial y_1} + x_2 \frac{\partial f(y_1, y_2, y_3)}{\partial y_2} + x_3 \frac{\partial f(y_1, y_2, y_3)}{\partial y_3} = 0,$$

FIG. 103.—SIMPLE POINT AND TANGENT

which has at least two intersections with C at (y_i).

This line is called the *tangent* to C at (y_i); if C meets it more than twice at the point, the point is called an *inflexion* and the tangent is called an *inflexional tangent*. In general, a simple point is not inflexional (Fig. 103).

Before continuing, it is desirable to quote Euler's theorem for a homogeneous polynomial. We require this in the forms

$$\Sigma y_i \frac{\partial f}{\partial y_i} \equiv nf,$$

$$\Sigma y_i \frac{\partial^{a+b+c+1} f}{\partial y_1^a \partial y_2^b \partial y_3^c \partial y_i} \equiv (n - a - b - c) \frac{\partial^{a+b+c} f}{\partial y_1^a \partial y_2^b \partial y_3^c},$$

where f stands for $f(y_1, y_2, y_3)$ and is of degree n.

By Euler's theorem, if all the $(r - 1)$th partial derivatives of $f(y_1, y_2, y_3)$ vanish, so do all those of less order and so does $f(y_1, y_2, y_3)$. Suppose that this is the case but that at least one rth partial derivative does not vanish. The equation in λ, μ becomes

$$\frac{\lambda^{n-r}\mu^r}{\underline{|r}} \nabla^r f(y_1, y_2, y_3) + \frac{\lambda^{n-r-1}\mu^{r+1}}{\underline{|r+1}} \nabla^{r+1} f(y_1, y_2, y_3) + \ldots = 0.$$

Every line through (y_i) is met by C in $n - r$ points different from (y_i), except in the case of the r lines (some of which may coincide), given by

$$\left(x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial y_3}\right)^r f(y_1, y_2, y_3) = 0.$$

That the last equation does represent a set of lines is easily proved by remarking that if (z_i) satisfies the equation, so does $(\theta y_i + \phi z_i)$; Euler's theorem is involved here.

Each of the r lines is met at least $r + 1$ times by C at (y_i) and is called a *tangent* at the point. The point (y_i) is said to have *multiplicity* r, or to be *r-ple*, for C; and is called *ordinary* when the r tangents are distinct. For $r = 2, 3$ we speak of *double* and *triple* points respectively; a double point with distinct tangents is called a *node*, one with coincident tangents a *cusp* (Fig. 104). A curve with only simple points is *non-singular*.

The number of $(r - 1)$th partial derivatives of $f(y_1, y_2, y_3)$ is $\frac{1}{2}r(r + 1)$. Therefore in constraining curves of order n ($\geqslant r$) to have a given point to multiplicity r, $\frac{1}{2}r(r + 1)$ independent linear

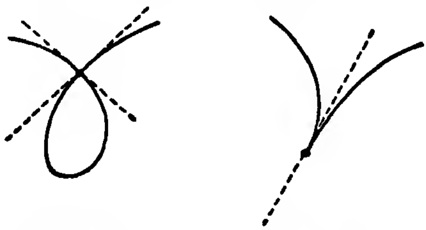


FIG. 104.—NODE WITH ITS TWO TANGENTS, AND CUSP WITH ITS ONE TANGENT.

conditions are imposed. If the r-ple point has to be of a particular type or have assigned tangents, further conditions are imposed which may or may not be linear.

Ex. 3. To examine the nature of a particular multiple point it is often convenient to take a triangle of reference with one vertex, say (0, 0, 1), at the point. If the point is r-ple, prove that the equation of the curve takes the form

$$z^{n-r}u_r + z^{n-r-1}u_{r+1} + \dots + u_n = 0,$$

where u_s is a homogeneous polynomial in x, y of degree s. The tangents at the multiple point are given by $u_r = 0$.

In particular, if the point is an ordinary double point, the equation may be taken in the form

$$z^{n-2}xy + z^{n-3}u_3 + \dots + u_n = 0;$$

and if it is a cusp, the equation may be taken to be

$$z^{n-2}y^2 + z^{n-3}u_3 + \dots + u_n = 0.$$

Ex. 4. The multiplicity of a point is a projective invariant of the curve.

(iv) Monoids.—A curve of order n which has an $(n-1)$-ple point O is called a *monoid*. It is clear that a necessary and sufficient condition for a monoid to be irreducible is that there should be no other multiple point. We prove that an irreducible monoid is rational.

Note, first of all, that a variable line through O meets the curve in one other point; the points of the curve are thus, except for O, in $(1, 1)$ correspondence with the parameters of the lines through O; we have to prove that the co-ordinates of a point on the curve are proportional to rational functions of the corresponding parameter.

Taking O to be $(0, 0, 1)$, the equation of the monoid has the form

$$zu_{n-1} + u_n = 0,$$

u_{n-1} and u_n having no common factor. Put $y = \lambda x$; then

$$zu_{n-1}(1, \lambda) + xu_n(1, \lambda) = 0.$$

The curve therefore has the regular parametric equations

$$x : y : z = u_{n-1}(1, \lambda) : \lambda u_{n-1}(1, \lambda) : -u_n(1, \lambda).$$

A particular case is that a cubic curve with just one double point is irreducible and rational.

(v) Order and rank of a branch.—We must now refer in more detail to the notion of branch, introduced at the beginning of section 57. For the moment we confine attention to an irreducible rational curve C.

Consider a point P (ξ_i) on the curve at which t_0 is one of the values taken by the regular parameter t. The equations of the curve may be expressed in the form

$$x_i - \xi_i = a_{i,1}(t - t_0) + a_{i,2}(t - t_0)^2 + \dots + a_{i,n}(t - t_0)^n, \qquad i = 1, 2, 3.$$

For values of t within a certain domain Δ including t_0, the equations represent the branch at the place t_0, this place is called the *origin* of the branch

The equation of a line through P has the form

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix} = 0.$$

It meets the branch at places given by the equation

$$0 = (\xi_2\lambda_3 - \xi_3\lambda_2)(a_{1,1}(t - t_0) + \dots + a_{1,n}(t - t_0)^n) + \text{two similar expressions.}$$

For general values of λ_1, λ_2, λ_3, the polynomial on the right hand side of this equation involves $t - t_0$ to a certain lowest power s. We call s the *order* of the branch, this having an s-ple point at P; if $s = 1$ we say that the branch is *linear*

The coefficient of this term in $(t - t_0)^s$ is linear in λ_1, λ_2, λ_3. If we choose λ_1, λ_2, λ_3 so that this coefficient is zero, the polynomial will involve $t - t_0$ to a new lowest power $s + s_1$; and, corresponding to this restriction on λ_1, λ_2, λ_3, we have a single line through P which meets the branch $s + s_1$ times at the place t_0 and is the only line through P to meet the branch more than s times at the place. We call s_1 the *rank* of the branch and this single line the tangent to the branch at P.

The preceding definitions require modification if the regular parameter t takes the value ∞ at the origin of the branch. In this case we choose a new parameter $t' = 1/t$ and frame the definitions relative to this new parameter, the origin of the branch being given now by $t' = 0$.

Ex. 5. Show that the quartic monoid $zx(x + y)^2 = x^4 + y^4$ has the parametric equations $x : y : z = (1 + t)^2 : t(1 + t)^2 : 1 + t^4$. The point $(0, 0, 1)$ is a triple point at which t takes the values $-1, \infty$. The branch with origin at $t = -1$ is of order 2, rank 1, and has $x + y = 0$ as tangent; the branch with origin at $t = \infty$ is linear, of rank 1, and has $x = 0$ as tangent.

Ex. 6. At a simple point of C there is just one branch and this is linear

Ex. 7 The order, rank and tangent of a branch are invariant relative to a projective change of parameter.

Ex 8. The order and rank of a branch are invariant relative to a projective transformation of the plane If, then, we choose a triangle of reference so that P has co-ordinates $(0, 0, 1)$ and the tangent has the equation $x_2 = 0$, the equations of the branch take the form

$$
\begin{aligned}
x_1 &= b_{1,s}(t - t_0)^s + \dots + b_{1,n}(t - t_0)^n,\\
x_2 &= b_{2,s+s_1}(t - t_0)^{s+s_1} + \dots + b_{2,n}(t - t_0)^n,\\
x_3 &= b_{3,0} + b_{3,1}(t - t_0) + b_{3,2}(t - t_0)^2 + \dots + b_{3,n}(t - t_0)^n,
\end{aligned}
$$

with $b_{1,s}$, $b_{2,s+s_1}$, $b_{3,0}$ all different from zero.

(vi) Quadratic plane transformations. Analysis of multiple points.

(*a*) The next step is to introduce a new kind of transformation, called *quadratic* on account of the quadratic polynomials involved; it gives considerable insight into the structure of multiple points.

Let x, y, z be co-ordinates in a plane π and x', y', z' be co-ordinates in a plane π'. The planes may coincide, and then also

the two systems of reference may coincide. The transformation T in question is given by

$$x' : y' : z' = yz : zx : xy.$$

These equations are reversible and lead to the inverse transformation T^{-1} given by

$$x : y : z = y'z' : z'x' : x'y',$$

which equations are of the same form as those of T.

Let ABC, $A'B'C'$ be the triangles of reference in π, π' respectively. To every point in π, not on the sides of ABC, corresponds a unique point in π' not on any side of $A'B'C'$, and *vice versa*; in this sense the transformation is said to be (1, 1). On the other hand A corresponds to the whole line $B'C'$ in T, and every point on $B'C'$ corresponds to A in T^{-1}. Similar remarks apply to the other vertices and sides of the two triangles. We say that A is a *fundamental point* of T, that $B'C'$ is a *fundamental line* of T^{-1}, and so on.

Now consider a line L through A, which we fix by joining A to the point F $(0, b, c)$ on BC. Every point P on L has co-ordinates of the form (λ, b, c); the corresponding point P' has the co-ordinates $(bc, c\lambda, b\lambda)$, and varies on the line L' joining A' to F' $(0, c, b)$. A, as a point of L, corresponds to F' on L', and *vice versa*. In this way, the points on $B'C'$ correspond projectively with the directions of the lines through A. We therefore widen the meaning of the word point and call the direction of AF at A a *neighbouring point* of A, the aggregate of these new points at A is called the (first) *neighbourhood* of A, which we denote by $[A]$. $B'C'$ may then be regarded as the transform of $[A]$ by T.

(*b*) Let C be a curve of order n in π, having points of multiplicities r, s, t at A, B, C respectively. Its equation may be arranged, first of all, in the form

$$x^{n-r}y^pz^qu_{r-p-q}(y, z) + x^{n-r-1}u_{r+1}(y, z) + \ldots + u_n(y, z) = 0,$$

where neither y nor z divides u_{r-p-q}. The points in π' which correspond to the points of C, other than those on the triangle ABC, are those, not on $A'B'C'$, whose co-ordinates satisfy the equation

$$(y'z')^{n-r}(z'x')^p(x'y')^qx'^{r-p-q}u_{r-p-q}(z', y') + \\ (y'z')^{n-r-1}x'^{r+1}u_{r+1}(z', y') + \ldots + x'^nu_n(z', y') = 0,$$

and *vice versa*. The left hand side of this equation contains the factor x'^r; by rearranging the form of the equation of C, this new equation is seen also to contain the factors y'^s, z'^t. The equation may therefore be written as

$$x'^ry'^sz'^t\phi_{2n-r-s-t}(x', y', z') = 0.$$

The points in question in π' are therefore just those which belong to the curve C′, whose equation is $\phi_{2n-r-s-t}(x', y', z') = 0$, but do not lie on $A'B'C'$. We call C′ the *reduced transform* of C relative to T.

The equation of C′ can clearly be written in the form

$$x'^{n-r}v_{n-s-t}(z', y') + \dots + v_{2n-r-s-t}(z', y') = 0,$$

and in two similar forms. Hence, C′ is a curve of order $2n - r - s - t$ having points of multiplicities $n - s - t$, $n - t - r$, $n - r - s$ at A', B', C' respectively.

From the identity

$$v_{2n-r-s-t}(z', y') \equiv y'^{n-r-s+q}z'^{n-r-t+p}u_{r-p-q}(z', y'),$$

we see that C′ meets $B'C'$ in $2n - r - s - t$ points, of which $n - r - t + p$ coincide at B', $n - r - s + q$ coincide at C', and there are $r - p - q$ other intersections (some of which may coincide) given by $x' = 0$, $u_{r-p-q}(z', y') = 0$. The $r - p - q$ points correspond to the $r - p - q$ points of $[A]$ defined by the tangents $u_{r-p-q}(y, z) = 0$ at A to C, each point being counted as often as the corresponding tangent. Of the $n - r - t + p$ points at B', p correspond to the p coincident points of $[A]$ defined by the tangents $y^p = 0$ at A to C; the other $n - r - t$ correspond to the intersections of C with $y = 0$ other than A, C; and $B'C'$ being a tangent to C′ at B' if $p > 0$.

If B, C are chosen in general positions relative to C, we have $p = q = s = t = 0$. The r tangents at A then give rise to r points on $B'C'$, not at B' or C'. If, at one of these, C′ has a point of multiplicity r_1, we say that C has an r_1-ple point at the corresponding point of $[A]$. Clearly $\Sigma r_1 \leqslant r$. The multiple points of C on $[A]$ are independent of the particular quadratic transformation of this type; they depend only on C.

Ex. 9. The multiple points of C on $[A]$ are projective invariants of C.

Ex. 10. The curve

$$x^{n-2}u_1^2 + x^{n-3}u_3 + \dots + u_n = 0$$

has a double point at A, both tangents coinciding with $u_1 = 0$. Prove that the curve has a double point on $[A]$, at the point associated with $u_1 = 0$, if and only if u_1 divides u_3. In this case the point A is called a *tacnode* (Fig. 105).

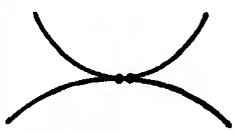


FIG. 105.—TACNODE.

(vii) Properties of a branch.—Now let C be irreducible and rational. The equations of a particular branch B, of order s and

rank s_1, at A, with tangent $y = \theta z$ ($\theta \neq 0, \infty$), have the form

$$x = 1 + a(t-t_0) + b(t-t_0)^2 + \ldots + c(t-t_0)^n,$$
$$y = \{d(t-t_0)^{s+s_1} + \ldots + e(t-t_0)^n\} + \theta\{f(t-t_0)^s + \ldots + g(t-t_0)^n\},$$
$$z = f(t-t_0)^s + \ldots + g(t-t_0)^n,$$

with $d \neq 0$, $f \neq 0$, and t in some domain Δ including t_0.

The reduced transform B′ of B has the equations

$$\begin{aligned} x' &= \theta f(t-t_0)^s + \text{higher powers of } (t-t_0), \\ y' &= \quad 1 \quad + \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \ , \\ z' &= \quad \theta \quad + \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \ . \end{aligned}$$

B′ therefore meets $B'C'$ s times at the point F' $(0, 1, \theta)$, which is the origin of B′.

If however B has $y = 0$ as tangent at A, the equations of B′ are

$$\begin{aligned} x' &= df(t-t_0)^{s+s_1} + \text{higher powers of } (t-t_0), \\ y' &= \quad f \quad + \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \ , \\ z' &= d(t-t_0)^{s_1} \quad + \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \ . \end{aligned}$$

B′ then meets $B'C'$ $s + s_1$ times at B', and $B'A'$ s_1 times at B'; hence B′ has an s_1-ple point at B'. A similar result is obtained if $z = 0$ is the tangent at A to B.

If, then, C has h branches at A, of orders $s^{(1)}, \ldots, s^{(h)}$, all with $y = \theta z$ as tangent, C′ meets $B'C'$ $s^{(1)} + \ldots + s^{(h)}$ times at F'. Therefore, if the equation of C is written in the form

$$x^{n-r}u_r + x^{n-r-1}u_{r+1} + \ldots + u_n = 0,$$

$y - \theta z$ occurs precisely $s^{(1)} + \ldots + s^{(h)}$ times as a factor of u_r.

And, if C has k branches at A, of orders $\sigma^{(1)}, \ldots, \sigma^{(k)}$ and ranks $\sigma_1^{(1)}, \ldots, \sigma_1^{(k)}$, all with $y = 0$ as tangent, C′ has a point of multiplicity $\sigma_1^{(1)} + \ldots + \sigma_1^{(k)}$ at B', and has $\sigma^{(1)} + \ldots + \sigma^{(k)}$ further intersections with $B'C'$ at B'. Again, therefore, y occurs precisely $\sigma^{(1)} + \ldots + \sigma^{(k)}$ times as a factor of u_r. A similar statement applies when $z = 0$ is a tangent to C at A.

Ex. 11. Verify these results in the cases of the curves mentioned in *Exs.* 5, 10.

Ex. 12. At an ordinary multiple point of a rational curve every branch is linear.

Ex. 13. C, D are two curves, possibly reducible but with no common component, of orders m, n respectively The common points are $O_1, \ldots, O_h$; at O_i C, D have multiplicities r_i, s_i respectively, but the tangents to C are different from the tangents to D there. Prove that, if all the components of C are rational, D meets C in mn places, of which $r_i s_i$ coincide at O_i; so that $mn = \Sigma r_i s_i$.

61. Some enumerative properties of rational curves.

(i) The class.—Let C be an irreducible rational curve of order n. We seek a formula for the number m of tangents which pass through a general point O, this number is the *class* of the envelope of tangents to C.

It is clear that, since O is in general position, each tangent through O has its point of contact at a simple point of C and meets C just twice there, and none touches C twice.

The equations of C being $x_i = f_i(t)$, the equation of the tangent at a simple point with parameter τ is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ f_1(\tau) & f_2(\tau) & f_3(\tau) \\ f_1'(\tau) & f_2'(\tau) & f_3'(\tau) \end{vmatrix} = 0.$$

Therefore, if $t_1, \ldots, t_m$ are the parameters of the points of contact of the tangents from O (y_i), the equation in t,

$$\phi(t) \equiv \begin{vmatrix} y_1 & y_2 & y_3 \\ f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \end{vmatrix} = 0,$$

has distinct roots $t_1, \ldots, t_m$ each occurring to multiplicity 1; and no other root corresponds to a simple point of C.

Now let A be a point at which C has a branch B of order s. We may suppose, without loss of generality, that the co-ordinate system has been chosen so that the equations of B are those given in part (vii) of section 60, with $\theta = 0$. Then $\phi(t)$ may be expressed as

$$\begin{vmatrix} y_1 & 1 + a(t-t_0) + \ldots & a + \ldots \\ y_2 & d(t-t_0)^{s+s_1} + \ldots & d(s+s_1)(t-t_0)^{s+s_1-1} + \ldots \\ y_3 & f(t-t_0)^s + \ldots & fs(t-t_0)^{s-1} + \ldots \end{vmatrix}$$

where the dots in each place denote higher powers of $(t - t_0)$. Thus $\phi(t)$ is expressible as a polynomial in $(t - t_0)$, the lowest power being $s - 1$. Therefore t_0 is a root of multiplicity $s - 1$ for $\phi(t) = 0$. Since $\phi(t)$ is of degree $2n - 2$ in t (it needs to be verified that the order is not $2n - 1$ as might appear at first sight) we therefore have

$$2n - 2 = m + \Sigma(s - 1),$$

the summation being extended over all places on C which are the origins of non-linear branches.

Ex. 1. The class of a nodal cubic curve is 4, of a cuspidal cubic is 3.

Ex. 2. Prove that the quartic curve given by

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+\frac{2f}{yz}+\frac{2g}{zx}+\frac{2h}{xy}=0$$

is rational and has, in general, nodes at the vertices A, B, C of the triangle of reference. Its class is 6.

In the particular case where the equation takes the form

$$\frac{p^2}{x^2}+\frac{q^2}{y^2}+\frac{r^2}{z^2}-\frac{2qr}{yz}-\frac{2rp}{zx}-\frac{2pq}{xy}=0,$$

the curve has cusps at A, B, C, the tangents there being concurrent. The class is 3.

Ex. 3. Referring again to a general rational curve C, a symmetric algebraic $(n-1, n-1)$ correspondence is seen to be set up on C by making two points correspond when they are in line with O. By considering the united places of this correspondence, evaluate m. These considerations are, in fact, another way of interpreting the algebra already used to evaluate m.

Ex. 4. The number of tangents which can be drawn through a general point of C to touch the curve elsewhere is $m-2$.

Ex. 5. If C has only ordinary multiple points, its class is $2n-2$, and the number of tangents which can be drawn from an r-ple point, at which each branch has rank 1, to touch the curve elsewhere is $2n-2-2r$.

(ii) Properties of a rational envelope.—The analysis of the structure of a curve may be applied dually to an envelope of lines. Consider then the envelope $\bar{\text{C}}$ consisting of the tangents to C; $\bar{\text{C}}$ is also rational. Defining a "*branch*" $\bar{\text{B}}$ in an obvious manner, its "*origin*" is a line L such that, of the lines of $\bar{\text{C}}$ through any point of L, i coincide with L, save for one position of the point for which $i+i_1$ coincide with L. The characters i, i_1 are dual to s, s_1 and are called respectively the *class* and *rank* of $\bar{\text{B}}$.

Consider now the "branch" $\bar{\text{B}}$ constituted by the tangents to the branch B referred to in part (i) of this section; the "origin" of $\bar{\text{B}}$ is the tangent to B at its origin.

The line-equations of $\bar{\text{B}}$ are easily seen to be

$$\begin{aligned} l_1 &= -fds_1(t-t_0)^{s+s_1} + \text{higher powers of } (t-t_0), \\ l_2 &= -fs \qquad\qquad\qquad\quad + \quad ,, \qquad ,, \quad ,, \quad ,, \quad , \\ l_3 &= d(s+s_1)(t-t_0)^{s_1} + \quad ,, \qquad ,, \quad ,, \quad ,, \quad . \end{aligned}$$

The "origin" of $\bar{B}$ is the line $x_2 = 0$; a general point on this has a set of co-ordinates $(1, 0, \alpha)$. The parameters of the lines of $\bar{B}$ which pass through this point are given by

$$\{-fds_1(t-t_0)^{s+n}+\ldots\}+\alpha\{d(s+s_1)(t-t_0)^n+\ldots\}=0.$$

This equation in t has t_0 as an s_1-ple root; therefore $i = s_1$. If, however, $\alpha = 0$, the point then being the origin of B, the equation has t_0 as an $(s + s_1)$-ple root; hence $i_1 = s$.

The dual of the class m of C is the number of points on a general line through each of which pass two coincident "*places*" of $\bar{C}$; these points are the intersections of the line with C and therefore are n in number.

Dualising the formula for the class of C, we thus obtain

$$\begin{aligned} n &= 2m - 2 - \Sigma(i-1) \\ &= 2m - 2 - \Sigma(s_1 - 1), \end{aligned}$$

the summation being extended over all branches of C of rank greater than 1.

Ex. 6. Prove that

$$3(n-m) = \Sigma(s - s_1), \quad 3(n-2) = \Sigma(2s + s_1 - 3),$$

the summations being extended over all branches.

FIG. 106—SIMPLE INFLEXION.

(iii) **Inflexions.**—An *inflexion* on a rational curve is defined to be the origin of a branch for which $s = 1, s_1 > 1$. If $s_1 = 2$, the inflexion is called *simple* (Fig. 106). For a rational curve this definition extends that given in part (iii) of section 60 to cover the possibility of the place being at a multiple point of the curve.

Ex. 7. Suppose that a rational curve has, as its only multiple points, δ nodes and κ cusps (each of rank 1), and further that its inflexions are all simple and ι in number. Then we have the simple formulae

$$\begin{aligned} m &= 2n - 2 - \kappa, \\ n &= 2m - 2 - \iota, \end{aligned}$$

which are equivalent to

$$\begin{aligned} 3(m-n) &= \iota - \kappa, \\ 3(n-2) &= 2\kappa + \iota, \\ 3(m-2) &= 2\iota + \kappa. \end{aligned}$$

Ex. 8. A nodal cubic has three inflexions. The equation of the cubic may be taken as $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 3yxz$. If $\theta = (a/d)^{\frac{1}{3}}$ and ω is a complex cube root of -1, prove that the inflexions are at the points $(1, -\theta, b - c\theta)$, $(1, \omega\theta, b + c\omega\theta)$, $(1, \omega^2\theta, b + c\omega^2\theta)$. These three points lie on the line $bx + cy = z$. By taking a new triangle of reference with this line as $z = 0$, the equation of the curve simplifies to $ax^3 + dy^3 = 3xyz$ or, if the unit point is chosen suitably, to $x^3 + y^3 = 3xyz$.

Ex. 9. A cuspidal cubic has one inflexion. Show that its equation may be put in the form $x^3 = y^2z$. The cusp is at $(0, 0, 1)$ and the tangent is $y = 0$; the inflexion is at $(0, 1, 0)$ and its tangent is $z = 0$.

(iv) First polar curves.—Let C be a rational or irrational irreducible curve, represented by the vanishing of the homogeneous polynomial $f(x, y, z)$, of degree n. $O(x_0, y_0, z_0)$ is a fixed point. The curve whose equation is

$$x_0\frac{\partial f}{\partial x} + y_0\frac{\partial f}{\partial y} + z_0\frac{\partial f}{\partial z} = 0$$

is called the *first polar curve* of O with respect to C; let us denote it by C′.

Suppose that O is in general position. Then the simple points of C which belong to C′ are the points of contact of the m tangents from O to C, where m is the class of C, defined for an irrational curve in the same way as for a rational curve.

C′ also passes through the multiple points of C; we now investigate in what manner.

First it is desirable to show that if a non-singular linear substitution is made in the co-ordinates, so that $f(x, y, z)$ becomes $F(X, Y, Z)$, and (x_0, y_0, z_0) becomes (X_0, Y_0, Z_0), then

$$x_0\frac{\partial f}{\partial x} + y_0\frac{\partial f}{\partial y} + z_0\frac{\partial f}{\partial z} \text{ becomes } X_0\frac{\partial F}{\partial X} + Y_0\frac{\partial F}{\partial Y} + Z_0\frac{\partial F}{\partial Z}.$$

Since, for all values of λ, μ, the triad $(\lambda x + \mu x_0, \lambda y + \mu y_0, \lambda z + \mu z_0)$ is transformed into $(\lambda X + \mu X_0, \lambda Y + \mu Y_0, \lambda Z + \mu Z_0)$, we have

$$f(\lambda x + \mu x_0, \lambda y + \mu y_0, \lambda z + \mu z_0) \equiv F(\lambda X + \mu X_0, \lambda Y + \mu Y_0, \lambda Z + \mu Z_0).$$

Expanding both polynomials and equating coefficients of $\lambda^{n-1}\mu$, we have the required result.

An interpretation of this lemma is that C′ is a projective covariant of the pair of entities, C, O.

To investigate the behaviour of C′ at a multiple point A of C, we change the triangle of reference so that A becomes the point

(1, 0, 0). If the multiplicity of A is r, the equation of C takes the form

$$x^{n-r}u_r(y, z) + x^{n-r-1}u_{r+1}(y, z) + \ldots + u_n(y, z) = 0.$$

According to the lemma, the equation of C′ is

$$\begin{aligned} & x_0[(n-r)x^{n-r-1}u_r(y, z) + \ldots + u_{n-1}(y, z)] \\ & + y_0\left[x^{n-r}\frac{\partial u_r}{\partial y} + \ldots\ldots\ldots\ldots\ldots + \frac{\partial u_n}{\partial y}\right] \\ & + z_0\left[x^{n-r}\frac{\partial u_r}{\partial z} + \ldots\ldots\ldots\ldots\ldots + \frac{\partial u_n}{\partial z}\right] = 0. \end{aligned}$$

Thus C′, of order $n - 1$, has an $(r - 1)$-ple point at A.

If u_r factorises into $(b_1y + c_1z)^{\mu_1} \ldots (b_ky + c_kz)^{\mu_k}$, where $\mu_1 + \ldots + \mu_k = r$, we have

$$\begin{aligned} \phi_{r-1}(y, z) &\equiv y_0\frac{\partial u_r}{\partial y} + z_0\frac{\partial u_r}{\partial z} \\ &= \Sigma\mu_i(b_iy_0 + c_iz_0)(b_1y + c_1z)^{\mu_1} \ldots (b_iy + c_iz)^{\mu_i - 1} \ldots (b_ky + c_kz)^{\mu_k}. \end{aligned}$$

The tangents to C′ at A are given by $\phi_{r-1}(y, z) = 0$. Since O is in general position, these tangents are $b_1y + c_1z = 0$, counted $\mu_1 - 1$ times, ..., $b_ky + c_kz = 0$, counted $\mu_k - 1$ times, together with $k - 1$ other lines not coincident with any of the first set.

Now restrict C to be rational. Then if A is an ordinary multiple point on C, we have $\mu_1 = \ldots = \mu_k = 1$, and the tangents to C′ at A are all different from the tangents to C there. Hence C′ meets C in exactly $r(r - 1)$ places at A. If, however, A is not an ordinary multiple point of C, at least one of the $\mu_i > 1$, and C′ meets C in more than $r(r - 1)$ places at A.

C′ meets C in $n(n - 1)$ places altogether, hence

$$n(n-1) = m + \Sigma r(r-1) + \epsilon,$$

where $\epsilon \geqslant 0$, equality holding if and only if all the multiple points of C are ordinary.

Limiting ourselves further to the case where the only multiple points of C are δ nodes and κ cusps (each of rank 1), it is clear that C′ meets C in two places at each node and in three at each cusp. Therefore

$$n(n-1) = m + 2\delta + 3\kappa,$$

and here $\epsilon = \kappa$.

Ex. 10. By dual considerations, prove that if the only "multiple lines" of the tangent envelope $\bar{\text{C}}$ are β bitangents of C and the tangents at ι simple inflexions, then $m(m - 1) =$

$n + 2\beta + 3\iota$. A *bitangent* of C is a line which touches the curve at two different places (Fig. 107).

Ex. 11. Prove that, if the only multiple points of C are δ nodes and κ cusps (of rank 1),

$$\tfrac{1}{2}(n-1)(n-2) = \delta + \kappa;$$

and, if the only "multiple lines" of $\bar{C}$ are β bitangents and ι simple inflexional tangents,

$$\tfrac{1}{2}(m-1)(m-2) = \beta + \iota$$

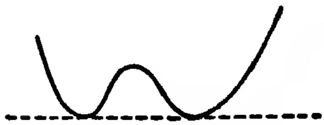


FIG. 107.—BITANGENT

62. Some general remarks and indications of further developments.

The main purpose of this chapter has been to describe a part of the theory of plane algebraic curves which is a natural sequel to the ideas which we have considered in some detail in regard to the geometry of lines and conics. Further, the aspects of geometry, which we have just dealt with, illustrate in a preliminary fashion the technique which is applied to the study of algebraic curves in general.

(i) The intersections of any two curves.—One of the principal advantages of limiting a discussion of general methods to the study of rational curves lies in the fact that it is a comparatively simple matter to attach a meaning to the term "intersections of a curve with a rational curve." It will have been apparent that much more could have been said about a general algebraic curve if we had settled the problems of how to interpret the "intersections of any two curves, rational or irrational" and of how to count the number of these intersections.

The success of the methods used here in regard to a rational curve depends upon the fact that the curve is representable by a set of parametric equations in which each value of the parameter is associated with one place on the curve, and *vice versa*.

It is important therefore to appreciate that the idea of parametric equations can be extended to any irreducible algebraic curve $f(x, y, z) = 0$. The essence of the matter is that, if (a, b, c) is an r-ple point of the curve, it is possible to find a set of equations of the form

$$\begin{aligned} x &= a + t^{s_i}, \\ y &= b + b_{i,1}t + b_{i,2}t^2 + \ldots, \\ z &= c, \\ i &= 1, 2, \ldots, k, \end{aligned}$$

such that, for $|t|$ small enough, the series for y is convergent and the points given by each group of equations belong to the curve. The set of points thus arising from any one group of equations

is called a *branch* or *cycle* of the curve at (a, b, c). The initial terms of the series for y are the most important in regard to the structure of the multiple point. We have moreover $s_1 + \ldots + s_k = r$; at a simple point of the curve $i = s_i = 1$.

We can thus, with small modifications, adapt to the general curve all that we have said about the branches of a rational curve and interpret and count in an obvious manner the "intersections" of another curve with the branch, and so with the whole of the curve $f(x, y, z) = 0$.

The results may be summed up for the moment by the limited statement that two curves of orders m, n, with no common component, have mn intersections; at a common point, r-ple for one and s-ple for the other, at which the two curves have different tangents, there are rs of the intersections; but if the two curves have at least one common tangent at the point then there are more than rs intersections there.

(ii) Cremona transformations.

(*a*) The quadratic transformation considered in section 60 (vi) is a particular example of a type of transformation called after *Cremona* (see H. P. Hudson's *Cremona Transformations* (Cambridge)).

Briefly, let

$$\lambda_1\phi_1(x_1, x_2, x_3) + \lambda_2\phi_2(x_1, x_2, x_3) + \lambda_3\phi_3(x_1, x_2, x_3) = 0$$

be the equation of an ∞^2 linear system (a *net*) of curves such that two curves of the system have just one variable common point. Put

$$y_1 : y_2 : y_3 = \phi_1 : \phi_2 : \phi_3,$$

(y_i) being the co-ordinates in another or the same plane.

To the curves of the net in the (x)-plane correspond the lines in the (y)-plane, and *vice versa*; and the variable point common to two curves of the net corresponds to the variable point common to the corresponding lines. There is thus determined a $(1, 1)$ correspondence between the points of the two planes, except for a finite number of fundamental points, which are common to all the curves of the net, and a finite number of fundamental curves, each of which has only fixed intersections with the curves of the net.

The equations of the correspondence are reversible in the form

$$x_1 : x_2 : x_3 = \psi_1(y_1, y_2, y_3) : \psi_2(y_1, y_2, y_3) : \psi_3(y_1, y_2, y_3),$$

where

$$\lambda_1\psi_1 + \lambda_2\psi_2 + \lambda_3\psi_3 = 0$$

is the equation of a similar net (such a net is called *homaloidal*)

of curves, with which is associated a set of fundamental curves and points in the second plane.

(*b*) The system of circles, in a modified complex euclidean plane, given by the equation

$$\lambda_1 xz + \lambda_2 yz + \lambda_3 r^{-2}(x^2 + y^2) = 0,$$

where λ_1, λ_2, λ_3 are parameters and $r(\neq 0, \infty)$ is a constant, form a homaloidal net whose fundamental points are at O (0, 0, 1) and the circular points I (1, i, 0), J (1, $-i$, 0) and whose fundamental curves are the lines OI, OJ, IJ. The net determines the quadratic transformation

$$x' : y' : z' = xz : yz : r^{-2}(x^2 + y^2),$$

these equations being reversed by

$$x : y : z = x'z' : y'z' : r^{-2}(x'^2 + y'^2).$$

If we suppose (x, y, z) and (x', y', z') to be co-ordinates of points P, P' in the same plane and referred to the same frame, the equations of the transformation assert that P' is the point common to the line OP and to the polar line of P with regard to the circle C

$$x^2 + y^2 = r^2 z^2;$$

and, by symmetry, P is derived in the same way from P'.

The points P, P' are said to be *inverse* with regard to C; and the quadratic transformation is described as *inversion relative to* C.

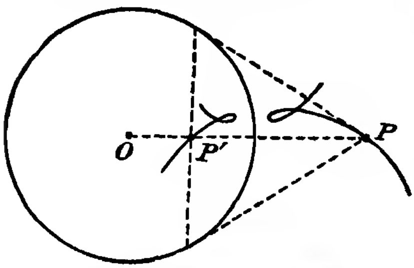


FIG. 108—INVERSE CURVES.

The curve F' which corresponds under this transformation to the curve F

$$f(x, y, z) = 0$$

has the equation

$$f(xz, yz, r^{-2}(x^2 + y^2)) = 0.$$

The reduced transform of F consists of the part of F' which remains after removing any fundamental components and its equation is obtained by removing from the last equation any factors of the form $(x + iy)^m$, $(x - iy)^n$, z^q. This reduced transform of F is called the *inverse curve* of F with regard to C (Fig. 108).

Ex. 1. The inverse of a general line is a circle through O.

Ex. 2. The inverse of a general line through O is the line itself.

Ex. 3. The inverse of a general conic is a quartic curve (a bicircular quartic) having nodes at O, I, J.

Ex. 4. The inverse of a general circle K is another circle belonging to the pencil determined by K, C.

Ex. 5. A circle orthogonal to C is its own inverse.

Ex. 6. A circle K through O inverts into the common chord of K, C.

Ex. 7. If two curves have r-point contact at a non-fundamental point P, the inverse curves have r-point contact at the corresponding point P'.

Ex. 8. The angles between two lines are equal to the angles of intersection of their inverse circles. Deduce that the angles between two curves at a common point P are equal to the angles between their inverses at the corresponding point P'.

Ex. 9. The inverse of a pencil of lines, with vertex V different from O, is a system of coaxal circles. The system of circles with centre at V inverts into the orthogonal coaxal system.

Ex. 10. The inverse of a general coaxal system of circles and of its orthogonal system is another coaxal system and its orthogonal system.

(iii) Neighbourhoods of a point. Clustered multiple points.— An important property of quadratic transformations is that by means of a finite number of these it is possible to transform a given curve into one having only *ordinary* multiple points, in particular into one having only nodes for multiple points.

The essence of the proof is at each stage to take one fundamental point A at one of the non-ordinary multiple points and the other two fundamental points B, C in general positions. This series of operations leads, moreover, to a characteristic description of the structure of any multiple point, which we indicate briefly.

Let A be an r-ple point of the curve C. Effecting a quadratic transformation as described, the corresponding reduced curve C′ may have a set of points A_1', A_2', ... of multiplicities r_1, r_2, ... on $B'C'$ but not at B', C'. Then $\Sigma r_i \leqslant r$. We have already decided to say that then C has multiple points A_1, A_2, ... of multiplicities r_1, r_2, ... in the first neighbourhood of A. Now take a new quadratic transformation with A_1' as a fundamental point. It then turns out that, in the first neighbourhood of A_1', C′ has a set of multiple points A_{11}', A_{12}', ... of multiplicities r_{11}, r_{12}, ..., with $\Sigma r_{1i} \leqslant r_1$. We say that C has points A_{11}, A_{12}, ... of multiplicities r_{11}, r_{12}, ... in the first neighbourhood of A_1, which is also called the *second neighbourhood of* A *relative to* A$_1$. Similar considerations apply to A_2, A_3, ...

Proceeding step by step we reach the idea of a *cluster* of multiple points in successive neighbourhoods of A; the operations are terminated when only simple points are reached. This cluster has the important property that for certain enumerative purposes, for example in counting the intersections of two curves, the

original multiple point at A may be regarded as replaceable by the cluster of multiple points each of which (including an ordinary r-ple point at A) may be regarded as an *ordinary* multiple point.

As an example of this, it may easily be verified that a curve C, which passes simply through a tacnode of a second curve D and has the same tangent there, has four intersections with D at the point.

(iv) The genus of a curve.—Suppose now that C is an irreducible curve of order n and that its multiple points have been replaced by equivalent clusters. Let r be the multiplicity of a typical multiple point. The number p, called the *genus* of C and defined by

$$p = \tfrac{1}{2}(n-1)(n-2) - \Sigma\tfrac{1}{2}r(r-1),$$

plays a basic part in the general theory regarding C. It is an integer, positive or zero. Not only is it a projective invariant of C, an invariant of C under Cremona transformations, but it is also invariant with respect to any *birational transformation* of C.

The last expression requires explanation. Two curves C, D are said to be in *birational correspondence* when their points are in (1, 1) correspondence in such a way that the co-ordinates of a general point of C are expressible (with the help of the equations of the two curves) as rational functions of the co-ordinates of the corresponding point on D, and *vice versa*. And in this respect, with suitable extensions of the definitions, the theorem of the invariance of p applies to curves in spaces of any number of dimensions.

The genus p thus serves to classify curves with respect to the wide class of birational transformations. In connection with the subject matter of this chapter, an especially interesting fact is that $p = 0$ is a necessary and sufficient condition for an irreducible curve to be rational The necessity may be proved in regard to a curve having only ordinary multiple points as follows.

By section 61 (i), we have for such a curve

$$2n - 2 = m;$$

and by section 61 (iv),

$$n(n-1) = m + \Sigma r(r-1).$$

Eliminating m, we have the required result at once.

The theory of the birational invariants of curves, surfaces and loci of higher dimension in space of any number of dimensions is fascinating. The literature is extensive; for a start, the reader who is interested should consult F. Severi, *Trattato di geometria algebrica*, Vol I, part 1, and H. F. Baker, *Principles of Geometry*, Vol. V.

MISCELLANEOUS EXERCISES

The following exercises are taken from examination papers set by the University of London.

The reader is asked to note these points:

(*a*) The cross-ratio symbol (A, B, C, D) usually means the same as our symbol $\{A, C; B, D\}$.

(*b*) Reciprocation with respect to a point O means reciprocation with respect to some circle with centre at O.

1. The pairs of points P_1, Q_1; P_2, Q_2; P_3, Q_3 on a straight line are given by the equations

$$S_i \equiv a_i x^2 + 2h_i x + b_i = 0, \ (i = 1, 2, 3)$$

where x is a parameter, not necessarily length, defining the position of a point. Prove that the members of the involution defined by the pairs P_2, Q_2 and P_3, Q_3 are given by the equation $S_2 + \lambda S_3 = 0$, and that the double points of the involution are given by

$$\begin{vmatrix} a_2 x + h_2 & h_2 x + b_2 \\ a_3 x + h_3 & h_3 x + b_3 \end{vmatrix} = 0.$$

Find the condition that the three given pairs should be in involution.

A, B, C are three given points on a straight line; A' is the harmonic conjugate of A with respect to B and C, and points B', C' are defined similarly. Prove that the pairs A, A'; B, B'; C, C' are in involution.

2. If projectivities ω, ω' on a line are called *permutable* when $\omega\omega' \equiv \omega'\omega$, prove that products and inverses of projectivities which are permutable with ω are themselves permutable with ω.

If τ is a fixed involution whose double points are M, N, prove that any projectivity which is permutable with τ must be either (i) a projectivity ω having M, N as united points, or (ii) an involution σ having M, N as a pair of mates. To which of these categories do products such as $\omega\omega'$, $\omega\sigma$, $\sigma\sigma'$ belong?

3. Prove that $(ABPQ) \barwedge (ABQR)$ if and only if

$$(P, R)/Q \equiv (A, B)/Q.$$

A, B, C are three points in line. Show that there is a unique projectivity ω_1 on AB which has A for its only united point and transforms B into C.

A second projectivity ω_2 on AB has C for its only united point and transforms A into D. Prove that the projectivity $\omega_2\omega_1$ is involutory if and only if A, D are harmonic with respect to B, C.

4 Six collinear points $O, P_1, P_2, P_3, P_4, P_5$ are given. Explain how, with the ruler only, you would construct the point P_6 on the line so that the cross-ratio $(OP_4P_5P_6)$ shall be equal to the cross-ratio $(OP_1P_2P_3)$.

A series of collinear points $O, P_1, P_2, \ldots$ is such that $(OP_rP_{r+1}P_{r+2}) = (OP_1P_2P_3)$ for all integral values of r. Taking

$$(OP_1P_2P_3) = \frac{OP_1 \cdot P_2P_3}{OP_3 \cdot P_2P_1} = k,$$

prove that

$$(OP_1P_2P_r) = \frac{k}{1 + k}\{1 - (-k)^{r-2}\}.$$

5. Prove that the locus of the intersection of corresponding rays of two projective (homographic) pencils with different vertices is in general a conic, and explain the nature of the locus when the pencils are in perspective.

A conic S touches the sides BC, CA, AB of a triangle ABC at L, M, N, and a

variable tangent to S meets these sides in X, Y, Z, respectively. X', Y', Z' are points on the sides such that B, C *harm.* X, X'; C, A *harm.* Y, Y'; A, B *harm.* Z, Z'. Prove that AX', BY', CZ' meet at a point whose locus is a straight line, which is the polar, with respect to S, of the intersection of AL, BM and CN.

6. ABC is a triangle inscribed in a conic, and an arbitrary line l meets the sides BC, CA, AB in points P, Q, R respectively. The polar of P with respect to the conic meets l in P', and points Q', R' are defined similarly. D is an arbitrary point of the conic, and the lines DP', DQ', DR' meet BC, CA, AB in points F, G, H respectively. Prove that F, G, H are collinear.

7. Define the term *one-one correspondence*, and state in what way it is special for an *involution*.

Prove that the straight lines joining the pairs of points of an involution of a given conic are concurrent

A, B are two points on a conic S, and C, D are two points not on S or on the line AB. The lines BC, AD meet the conic in points X, Y respectively, and the lines CA, BD meet the conic in points L, M respectively. Prove that the lines XY, LM meet on CD.

8. Two lines l, m in the plane of a conic k and not conjugate with regard to k meet at a point O not on k. If P, Q lie on k and OP, OQ are mates in the involution pencil with double rays l, m, prove that the line PQ always touches a certain conic s, having for tangents l, m and the tangents to k at its intersections with l and m.

$ABCD$ is a simple quadrilateral whose four vertices lie on a conic t and whose sides AB, BC, CD, DA touch another conic s The diagonals AC, BD meet at O. Prove that there exists an infinity of such quadrilaterals $A'B'C'D'$, whose diagonals $A'C'$, $B'D'$ meet in the fixed point O, and which are inscribed in t and circumscribed about s.

9. Tangents are drawn to a conic from three points A, B, C Those from A meet BC in D, D', those from B meet CA in E, E'; and those from C meet AB in F, F' Prove that D, D', E, E', F, F' lie on a conic

10 A, B, C, D are four fixed points of a given conic, and P is a variable point of the conic. Prove that the cross-ratio $P(A, B, C, D)$ is constant.

Prove also that this cross-ratio is equal to that of the four points in which a variable tangent to the conic is cut by the tangents at A, B, C, D

11 If ω, σ are projectivities on a line, show that $\omega\sigma$ is an involution if $\sigma\omega$ is an involution. If σ is an involution, prove that $\omega\sigma$ is an involution if and only if the united points in ω correspond in σ.

A variable triangle PQR, inscribed in a conic s, has the side QR touching a conic s' which has double contact with s If PQ meets the chord of contact of s, s' at a fixed point, prove that PR passes through one or other of two fixed points on the same chord of contact.

12. Two chords AB, CD of a conic k meet at O; and I, J are the points of contact of k with its tangents from O Prove that the conic s through B, C, D which touches IA at A also touches IB at B, and that k is the locus of points (P) for which IP, JP are conjugate for s

Hence show that s touches JC, JD at C, D.

13 Two conics touch one another at D and from a variable point T of the common tangent at D the remaining tangents TP, TQ are drawn Prove that the envelope of a line TR, such that the cross-ratio of the pencil $T(DPQR)$ has a constant value, is in general a conic touching the other two conics at D, and also touching their remaining common tangents

14. The sides BC, CA, AB of a triangle are met at D, E, F by the lines joining a point P to the opposite vertices; and L, M, N are the intersections of BC, CA, AB with EF, FD, DE respectively. The tangential equations in any system of homogeneous co-ordinates, of the points A, B, C are $A = 0$, $B = 0$, $C = 0$, so adjusted that $A + B + C = 0$ is the equation of the point P. Find, in terms of A, B, C, the tangential equations of the points D, E, F, L, M, N.

Obtain the tangential equation of the conic which touches AB, AC, PB, PC, MN, in the form

$$A(A + B + C) - 3BC = 0,$$

and show that the polar of P with respect to this conic passes through L.

If X, Y, Z are the points of contact of the line LMN with the three proper conics, each of which also touches two sides of the triangle ABC and two of the lines PA, PB, PC, prove that $(LMNXYZ) \barwedge (MNLYZX)$.

15. Two coplanar conics s, t are met by a line in pairs of points A, B and C, D respectively Prove that the four points in which the tangents to s at A, B are met by the tangents to t at C, D lie on the same conic k of the pencil determined by s and t

Hence show that if X, Y, Z, T be any four points of a conic k, and s be a conic touching XY and ZT at points A, B respectively, then YT and ZX are tangents, at their intersections with AB, to the same conic t of the pencil determined by s and k

State the plane duals of the above theorems

16. ABC, PQR are two triangles inscribed in a conic; prove that the points of intersection (BR, CQ), (CP, AR), (AQ, BP) lie on a line l (Pascal's Theorem)

If A, B, C are fixed, and P, Q, R vary so that AP, BQ, CR meet on a given line m, prove that l passes through a fixed point

If A, B, C are fixed, and P, Q, R vary so that AP, BQ, CR meet on a given conic Ω through A, B, C, prove that l envelops a conic

17. State and prove Pascal's theorem concerning a hexagon inscribed in a conic

P is a point on the circumcircle of a triangle ABC, and H is any point in the plane of the triangle. BH, CH cut the circumcircle at Y, Z and PY, PZ cut CA, AB respectively at V, W Prove that V, H, W are collinear

Deduce the theorem that the Simson's line of a point P on the circumcircle of the triangle ABC bisects the line joining P to the orthocentre.

18 Two points, one on each of two fixed straight lines, are conjugate with respect to a fixed conic Show that, in general, the envelope of the straight line joining them is a conic which touches the given straight lines and the tangents to the given conic at its points of intersection with the straight lines

Under what circumstances does the envelope degenerate into two points?

The chords AA', PP', QQ' of a conic are concurrent Prove that PQ, $P'Q'$, PQ', $P'Q$, AA' and the tangents at A and A' touch a conic

19. Show that the conics through four points determine an involution on a general line in their plane.

A fixed conic k passes through one vertex A of a quadrangle $ABCD$, but not through the other three vertices, and is met again by a variable conic of the pencil through A, B, C, D at points X, Y, Z Prove that the sides of the triangle XYZ touch a fixed conic inscribed in the triangle BCD

What form does this theorem take when C, D are the circular points?

20 Prove that the lines joining the point $P(x_1, y_1, z_1)$ to the intersections of the conic $S = 0$ with the line $L = 0$ are given by the equation

$$SL_1^2 - 2S_1LL_1 + S_{11}L^2 = 0,$$

with the usual notation.

21. Prove that the locus of points of contact of the tangents drawn from an arbitrary fixed point P to the conics circumscribed about a given quadrangle $ABCD$ passes through, P, A, B, C, D, through the vertices E, F, G of the diagonal triangle of the quadrangle, and through the intersections of PE, PF, PG with FG, GE, EF respectively

The equations of AB, BC, CD, DA being respectively $R = 0$, $S = 0$, $T = 0$, $U = 0$, prove that this locus is the cubic curve given by $R_1/R - S_1/S + T_1/T - U_1/U = 0$, where R_1, S_1, T_1, U_1 are the values of R, S, T, U at P. Show also that this locus touches at P the conic $ABCDP$.

22. $P_i = 0$ $(i = 1, 2, 3, 4, 5, 6)$ are tangential equations of six coplanar points. Prove that these six points lie on a conic if and only if there is an identity of the form

$$\lambda_1 P_1^2 + \lambda_2 P_2^2 + \ldots + \lambda_6 P_6^2 \equiv 0,$$

where (λ_i) are constants.

Hence, or otherwise, show that, if two triangles are inscribed in a conic, there exists a conic with respect to which they are both self-polar.

23. (i) Show that the frame of reference may be so chosen that two given conics with four distinct common points have the equations

$$xy = z^2 \text{ and } kxy = (x + y + cz)^2.$$

(ii) The tangents to a conic t from a variable point P of a conic s meet s again at Q and R. Prove that QR touches a third fixed conic k. Show also that the tangent to s at its second intersection with the tangent to t at a common point of s, t is a tangent to k; and that the second tangent to t from the point of contact of s with a common tangent of s, t is also a tangent to k.

24. A triangle $A_1A_2A_3$ is inscribed in a conic s. Prove that there is one non-degenerate conic t_i which touches A_iA_j, A_iA_k at A_j, A_k respectively, and also touches s $(i, j, k$ being any permutation of $1, 2, 3)$

If B_i is the point of contact of t_i with s, show that the three lines (A_iB_i) concur at a point O, and that the polars of O with respect to the three conics (t_i) meet in pairs on the lines (A_iB_i)

25 Show that, by a proper choice of general homogeneous co-ordinates, the equations of any four straight lines, in general position, may be taken to be $x \pm y \pm z = 0$.

If the conic $ax^2 + by^2 + cz^2 = 0$ touches these lines, prove that any tangent to it meets the lines in four points whose cross-ratio is the ratio, with the sign changed, of two of the three quantities a, b, c

26. Prove that the equation of the tangent at the point (x_1, y_1, z_1) to the conic whose equation is

$$f/x + g/y + h/z = 0$$

can be reduced to the form

$$\frac{fx}{x_1^2} + \frac{gy}{y_1^2} + \frac{hz}{z_1^2} = 0.$$

A conic s is inscribed in the triangle XYZ. Prove that among the proper conics which pass through X, Y, Z there is one which has double contact with s, and there are three which touch s and pass through a given point of general position in the plane

27. Prove that the equation of a conic inscribed in the triangle of reference XYZ of general homogeneous co-ordinates can be expressed in the form

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0.$$

Prove that the six points of contact with the sides of the triangle XYZ of two such conics S_1, S_2 lie on a conic which also passes through the two points of contact of the fourth common tangent t, of S_1S_2.

If t meets the sides YZ, ZX, XY in P, Q, R respectively, prove that the four lines which are the polars of X and P with respect to S_1 and S_2 meet in a point L; and that, if two points M, N are defined similarly, then XL, YM, ZN are concurrent.

28. If $\Sigma = 0$, $\alpha = 0$, $\beta = 0$ are the tangential equations of a conic and two points, interpret geometrically the equations

$$\Sigma + \alpha\beta = 0, \quad \Sigma + \alpha^2 = 0.$$

Each of two conics Σ_1 and Σ_2 has double contact with a third conic Σ. Prove that two of the points of intersection of the common tangents of Σ_1 and Σ_2 lie on the line joining the poles of the chords of contact of Σ_1 and Σ_2 with Σ, and form with them a harmonic range.

29. Prove that the projectivity ω, which transforms three real points A, B, C of a line l into B, C, A respectively, associates the points of l in triads permuted cyclically by ω; and that there is no real point self-corresponding in ω.

Two points U, V are coplanar with l, and k is the conic generated by the related flat pencils in which the triad $U(ABC)$ corresponds to $V(BCA)$; prove that the triads of successive homologues in ω are projected from U, or V, into triads of points of k which are sets of successive homologues in a definite projectivity on k, having l for its cross-axis

30. Four points O, A, B, C are of general position in a plane. Prove that all the conics (s), having AB, AC for tangents and BC for the polar of O, have two other fixed tangents

Show that a general point P in the plane lies on two of these conics (s), and that the tangents at P to these two conics meet BC at points harmonically conjugate with respect to B, C.

Interpret these results metrically when B and C are the circular points

31. Prove that if four conics have four distinct common points, they have just one common self-polar triangle, and their common tangents meet in pairs on the sides of this triangle

Two conics s, s' intersect at given points A, B and touch two given lines c, d. The line joining their other two common points meets AB at V, and their other two common tangents meet at W. Show that either V, W are in line with $O \equiv c \, . \, d$, or else OV, OW are the double rays of the involution pencil in which (OA, OB) and (c, d) are two pairs; and that the second case occurs if the poles of AB with respect to s and s' lie on OW.

State the special form of this theorem for two conics through A and B with a common focus O

32. Prove that the diagonal triangle of a quadrangle $ABCD$ inscribed in a conic is self polar with respect to the conic, and is also the diagonal triangle of the quadrilateral of tangents at A, B, C, D.

If A', B' are mates in the involution ω_1 in which (A, B) and (C, D) are pairs, and A', C' are mates in the involution ω_2 in which (A, C) and (B, D) are pairs, prove that the mate D' of B' in ω_2 is also the mate of C' in ω_1; and show that $(A, D), (B, C), (A', D'), (B', C')$ are four pairs in involution.

Hence show that if PQ, RS are two parallel chords of a hyperbola, and the parallels to the asymptotes through the point $PS \, . \, QR$ meet the hyperbola in accessible points T, U, then TU is parallel to PQ and RS

33. A, B, C, D are four points in a plane, no three being in line. If x, y, z are the co-ordinates of any point with ABC as triangle of reference and D as unit point, while X, Y, Z are the co-ordinates of the same point with ABD as triangle of reference and C as unit point, prove that

$$X : Y \cdot Z = z - x : z - y : z.$$

The conic d has ABC as a self-polar triangle and D as the pole of a given line r not passing through any of A, B, C, D; and a, b, c are three other conics determined in a similar way, by cyclic permutation of A, B, C, D. Prove that c, d meet twice on AB, and that a, b, c, d have in common two points on r.

Interpret these results when D is the orthocentre of the triangle ABC and r is the line at infinity.

34. Discuss the metrical character of a projectivity on a line in the cases when the point at infinity is (a) one of the two united points, (b) the only united point in the correspondence.

A given conic s has a fixed diameter AB, and O is a fixed point on the tangent at A. If a variable chord of s is drawn through O, prove that the lines joining A to its extremities intercept on the tangent at B a segment whose middle point is fixed.

35. Explain how to find the asymptotes of a conic when its equation is given in areal co-ordinates.

If P is the point (X, Y, Z) in areal co-ordinates show that the tangential

equation of the curve enveloped by the asymptotes of the conics through P and the vertices of the triangle of reference is

$$\frac{(m-n)^2 l}{X} + \frac{(n-l)^2 m}{Y} + \frac{(l-m)^2 n}{Z} = 0.$$

The projectivity ω which transforms A, A', B into A', A'', B' also transforms B' into B''. Prove that the pairs (A, B), (A', B'), (A'', B'') are in involution if, and only if, A', B' are harmonic with respect to A, A''.

Show that, when this condition is satisfied, any pair in the involution determined by (A, B) and (A', B') is transformed by ω into a pair in the same involution.

36. Prove that the director circles of the conics of a range form a pencil.

Hence, or otherwise, show that (i) the middle points of the diagonals of a complete quadrilateral are in line, (ii) the orthocentre of a triangle circumscribed about a parabola is on the directrix.

37. Prove that the locus of the intersection of corresponding rays of two homographic (projective) pencils is a conic passing through the vertices of the pencils.

a and b are two fixed diameters of a conic. P is any point such that the tangents from O to the conic are harmonically conjugate with respect to the lines through P parallel to a and b. Show that the locus of P is, in general, a hyperbola having a and b as asymptotes and passing through the extremities of the diameters conjugate to a and b

If a and b are perpendicular, prove that the hyperbola passes through the foci of the given conic

38. Any two straight lines through a fixed point O meet a given straight line t in P, Q, and P', Q' are the mates of P, Q in a given involution on t. Prove that the straight lines through P' and Q' parallel to OQ and OP respectively meet on the line joining O to the centre of the involution.

L, M are any two points on a given chord l of a conic. The tangents from L and M to the conic meet a fixed tangent t in P, P' and Q, Q' respectively, and the tangents at the points in which l meets the conic meet t in A and B. O is a fixed point not lying on t The straight lines through P' and Q' parallel to OQ and OP respectively meet in R. Prove that R lies on the straight line joining O to the midpoint of AB.

39. A variable tangent t meets four fixed tangents a, b, c, d of a conic S in points A, B, C, D respectively. Prove that the cross-ratio (A, B, C, D) is independent of t.

The triangle PQR circumscribes a parabola, and QR is parallel to the polar of P. Prove that, if another tangent of the parabola meets QR, RP, PQ in L, M, N respectively, then L is the middle point of MN.

40. A conic s is circumscribed about a triangle ABC, and a line p is met by s at Q and R, and met at J, K, L, T by BC, CA, AB and the tangent at A respectively. Prove that $(QJKL) \barwedge (RTLK)$, and show that, if p moves so that $\{QJKL)$ has a given constant value, R remains fixed.

Through any point Q of a hyperbola s a line p is drawn so that Q bisects the segment intercepted on p by the parallels to the asymptotes through a fixed point A of s Prove that p always passes through the other extremity of the diameter through A.

41. Show that the foci of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are given by the equations

$$\frac{\left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2}{a-b} = \frac{\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}}{h} = 4S.$$

Conics are drawn having double contact with a given conic at the ends of a

fixed chord parallel to one of its axes. Show that their foci lie on the other axis, or on a fixed circle with its centre on this axis.

42. Find the condition that the lines $y = mx$, $y = m'x$ should be parallel to conjugate diameters of the (central) conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If $X \equiv ax + hy + g$, $Y \equiv hx + by + f$, show that, for any value of λ, the equation

$$(hX^2 - aXY) + \lambda(bXY - hY^2) = 0$$

represents a pair of conjugate diameters of the conic.

Prove that the equation represents the axes of the conic if

$$\lambda(h - b\cos\omega) = (h - a\cos\omega),$$

where ω is the angle between the axes of co-ordinates.

43. Prove that the conic $ax^2 + 2hxy + by^2 = 1$ has the axes

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

Prove also that, if two central conics have their axes parallel, their points of intersection are concyclic.

44 Prove that the straight lines $ax^2 + 2hxy + by^2 = 0$ are parallel to conjugate diameters of the central conic $a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ if $ab' + a'b = 2hh'$.

P and Q are two points on a conic s', whose centre is O, such that OP and OQ are parallel to conjugate diameters of another central conic s Prove that the envelope of PQ is, in general, a conic having its asymptotes parallel to those of s.

If, however, the asymptotes of s' are parallel to conjugate diameters of s, prove that PQ is parallel to one of the asymptotes of s.

45. If P is a fixed point of a conic k, and Q, R any two points of k such that PQ, PR are conjugate with respect to a second conic c, prove that QR passes through a fixed point F; and that the polar of F with respect to k intersects the tangents from P to c in points of k.

Show also that, if c is a point-pair (A, B), the locus of F when P describes k is, in general, a conic which touches k at its intersections with the line AB

Hence, or otherwise, show that the locus of the Frégier points of a parabola is an equal parabola, with the same axis; the distance between the vertices being equal to the latus rectum.

46 If $\Sigma = 0$ is the tangential equation of a conic, and $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, are the equations of points, interpret the equation $\Sigma + k\alpha\beta = 0$, $\Sigma + k\alpha^2 = 0$, $\alpha\beta + k\gamma\delta = 0$, where k is constant.

P_1, P_2, Q_1, Q_2 are four points on an ellipse whose centre is O; R_1 and R_2 are the poles of the chords P_1Q_1 and P_2Q_2 respectively. Prove that the six sides of the triangles $P_1Q_1R_1$ and $P_2Q_2R_2$ touch a conic.

Show also, that if OR_1 and OR_2 are conjugate diameters of the ellipse, this conic is a parabola, and that its axis is parallel to the line joining O to the middle point of R_1R_2.

47. Prove that the lines joining a point P to the pairs of opposite vertices XX', YY', ZZ' of a quadrilateral form a pencil in involution, whose double lines pass through the intersections of ZZ' with the conic through X, X', Y, Y', and P.

Two hyperbolas S, S' have parallel asymptotes. $AA'BB'$ is a parallelogram whose sides are parallel to the asymptotes and whose opposite vertices A, B lie on S, and A', B' on S'. AB meets S' in K', L' and $A'B'$, meets S in K, L. Show that KK', LL' meet at one intersection of S and S', and KL', $K'L$ at the other.

48 Show that lines joining corresponding points of two projective (homographic) ranges on coplanar lines touch a conic.

A variable line through a fixed point O cuts a fixed line at P, and PQ is drawn making a fixed angle (in a given sense) with OP. Prove that PQ touches a parabola.

Show how to determine (i) the direction of the axis of the parabola; (ii) the tangent at the vertex.

49. Parabolas are drawn touching the sides of a triangle ABC. Prove that the points of contact of tangents to these parabolas that are parallel to a given line lie on the parabola passing through the vertices of the triangle ABC and having its axis parallel to the given line

50. Prove that the co-ordinates of the centre of curvature of the conic

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$$

at the point (ξ, η) are

$$\frac{\alpha - \beta}{\alpha^2}\xi^3, \quad \frac{\beta - \alpha}{\beta^2}\eta^3.$$

If C_1, C_2 are the centres of curvature at a point P of the two conics through P confocal with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

prove that C_1C_2 is a diameter of these conics if and only if P lies on the hyperbola

$$x^2 - y^2 = a^2 - b^2.$$

51. If the Cartesian axes of reference are inclined at an angle ω, find the equation of the circle with the points (x_1, y_1), (x_2, y_2) as ends of a diameter.

A chord HK of a hyperbola passes through a fixed point and the circle on HK as diameter meets the hyperbola again in L and M. Prove that LM also passes through a fixed point.

52. If $T = 0$ is the equation of a tangent at a point P of a conic $S = 0$, and $L = 0$ is the equation of any other line through P, interpret the equations

$$S - kT^2 = 0, \quad S - kTL = 0, \quad S - kL^2 = 0,$$

where k is an arbitrary constant.

Find the locus of the centres of rectangular hyperbolas which have four-point contact with a given parabola.

53. Prove that the polars of a point P with respect to the conics through four points meet in a point P', and that, when P describes a line l, P' describes a conic through the vertices of the common self-polar triangle of the conics. Show that this conic is also the locus of the poles of the line l with respect to the conics of the pencil.

The asymptotes of a conic s of the pencil meet the conic k which is the locus of centres of the conics of the pencil in the centre C of s and in two further points X, Y. Prove that when s varies in the pencil the line XY turns about a fixed point.

Prove a similar result for the principal axes of the conic s.

54. In a system of homogeneous co-ordinates the tangential equation of the circular points is

$$\Sigma \equiv al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm$$

A, B, C, F, G, H are the first minors of the corresponding small letters in the discriminant of Σ. Assuming that none of these first minors vanish, show that the equation of the line at infinity is

$$\frac{x}{F} + \frac{y}{G} + \frac{z}{H} = 0.$$

Find the condition that the conic $uyz + vzx + wxy = 0$ should be a rectangular hyperbola, and deduce that the co-ordinates of the orthocentre of the triangle of references are (f^{-1}, g^{-1}, h^{-1}).

55. Find the equation of the two straight lines through the origin perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$.

Two conics S_1, S_2 are so related that the asymptotes of S_1 are perpendicular

to those of S_2 (their centres being in general distinct). Prove that the locus of the centres of conics through the four common points of S_1, S_2 is a rectangular hyperbola.

BE, CF are two altitudes of a triangle ABC. Prove that two parabolas pass through the points B, C, E, F and that their axes are perpendicular.

56. Prove that the equation of a proper conic can be expressed in the form

$$2y = ax^2 + 2hxy + by^2.$$

Prove that the chords of a conic which subtend a right angle at a given point O of the conic all pass through a certain fixed point F.

Prove that, if F is the centre of the circle of curvature at O, then F is also the middle point of the chord of the conic which is normal at O.

57. Show that the tangential equation, in rectangular Cartesian co-ordinates, of any rectangular hyperbola with centre (x_0, y_0) can be written in the form

$$(lx_0 + my_0 + n)^2 + (l^2 - m^2) + \beta lm = 0.$$

Find necessary and sufficient conditions for the tangents at the points of parameters t_1, t_2, t_3, t_4, on the parabola k given by $x = at^2$, $y = 2at$ to be tangents to a rectangular hyperbola with its centre at the vertex O of k.

If P, Q are opposite vertices of the quadrilateral of common tangents of k and a rectangular hyperbola with centre O, prove that PQ is bisected by the axis of k, and that P, Q are conjugate points with respect to the circle which touches k at O and has its centre on the directrix of k.

58. The four normals at the ends of two chords of a conic are concurrent at a point O. Show that these ends lie on a rectangular hyperbola, passing through O and the centre of the conic, whose asymptotes are parallel to the axes of the conic.

Show that there are two positions of O such that one chord passes through a given point P and the other chord through a given point Q.

Prove that, if P is kept fixed and the two positions of O coincide, the locus of Q is a parabola touching the axes of the conic.

59. Prove that a general point in the plane of a conic k, given by the equation $ax^2 + by^2 = 1$, lies on four normals to k, and that, if the line $lx + my + n = 0$ joins the feet of two of these normals, the feet of the other two lie on the line $ax/l + by/m - 1/n = 0$.

If Q, R, S are the feet of the other three normals to k from a variable point X on the normal at a fixed point P, prove that the sides of the triangle QRS are tangents to a fixed parabola, which touches the axes of k and the lines joining the vertices of k on each axis to the image of P in that axis.

60. Show that, in general, four normals can be drawn to a central conic from a point in its plane.

Show that real parabolas can be drawn through the feet of the normals from a point to a real central conic if, and only if, the conic is an ellipse, and that in this case the axes of the parabolas are parallel to the equal conjugate diameters of the ellipse.

61. Find the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a \cos \alpha, b \sin \alpha)$.

Prove that the locus of the mid-points of normal chords of this ellipse is

$$\left(\frac{x^2}{a^6} + \frac{y^2}{b^6}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2y^2}{a^2b^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2.$$

62. Prove that the feet of the normals drawn from the point (h, k) to the ellipse $x^2/a^2 + y^2/b^2 = 1$ lie on the rectangular hyperbola

$$(a^2 - b^2)xy + b^2kx - a^2hy = 0.$$

PP' is a diameter of an ellipse whose centre is C, and Y is the foot of the perpendicular from C on the tangent at P. A circle through P and Y meets the ellipse again at Q, R, S. Prove that the normals at P', Q, R, S are concurrent.

63. The normals at four points on the ellipse $x^2/a^2 + y^2/b^2 = 1$ are concurrent at the point (h, k). Show that the normals at the corresponding points on any confocal ellipse are also concurrent, and that, if the confocal ellipse varies, the locus of the point of concurrence is the curve

$$\frac{h^2a^2}{x^2} - \frac{k^2b^2}{y^2} = a^2 - b^2.$$

(Points (x_1, y_1), (x_2, y_2) on the ellipses $x^2/a_1^2 + y^2/b_1^2 = 1$, $x^2/a_2^2 + y^2/b_2^2 = 1$ correspond if $x_1/a_1 = x_2/a_2$ and $y_1/b_1 = y_2/b_2$)

64. The normal at the point $P(ct, c/t)$ of the rectangular hyperbola $xy = c^2$ meets the hyperbola again in Q, and the circle on PQ as diameter meets the hyperbola in the point R different from P and Q. Prove that PR is a diameter of the hyperbola.

(i) The line QR meets the axes of co-ordinates in points M, N. Prove that, if Q and R are the points of trisection of MN, then $t^4 = \frac{1}{2}$.

(ii) Prove that, if the circle on QR as diameter touches the axis $y = 0$, then $t^4 = \frac{1}{3}$.

65. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$

Normals are drawn from the point $(am^2, 2am)$ to the parabola $y^2 = 4ax$. Show that the parameters t_1, t_2 of their feet $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ are the roots of the equation $t^2 + mt + 2 = 0$

Show that the product of their lengths is

$$4a^2(1 + m^2)^{3/2}.$$

66. Show that, if the conic whose trilinear equation is

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

is a rectangular hyperbola, then

$$a + b + c - 2f \cos A - 2g \cos B - 2h \cos C = 0,$$

where A, B, C are the angles of the triangle of reference

Show that the locus of the centres of rectangular hyperbolas touching three given straight lines is a conic

67. Find the condition that the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$ may touch the conic

$$a\sqrt{(f\alpha)} \pm b\sqrt{(g\beta)} \pm c\sqrt{(h\gamma)} = 0.$$

The co-ordinates being trilinear and a, b, c being the lengths of the sides of the triangle of reference, prove that, if the conic is a parabola, the point at infinity on the axis and the focus are respectively (f, g, h) and $(1/f, 1/g, 1/h)$

68. Obtain the tangential equations in trilinear co-ordinates of all conics which have two given points as foci.

A conic touches three given lines and one real focus lies on another given line Show that the locus of the other real focus is a conic passing through the intersections of the three given lines.

69. Show that the conic whose equation in areal co-ordinates is

$$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0.$$

is an ellipse if $fgh(f + g + h) > 0$.

Determine the parts of the plane of a triangle within which the centre of an ellipse touching the sides of the triangle must lie Illustrate your answer by means of a diagram, indicating the parts you have determined by shading.

70. Prove that the straight line $r \cos (\theta - \alpha) = p$ touches the parabola $l/r = 1 + \cos \theta$ if $p = \frac{1}{2}l \sec \alpha$

A circle passes through the focus S and has its centre on the axis of a fixed parabola. An ellipse with S as focus and having a tangent to the parabola as corresponding directrix meets in the circle in A, B, C, D. Prove that

$$SA + SB + SC + SD = 2la/l',$$

where a is the radius of the circle, $2l$ is the latus rectum of the parabola, and $2l'$ is the latus rectum of the ellipse.

71. Prove that the normal to the conic $l/r = 1 + e \cos \theta$ at the point for which $\theta = a$ is

$$e \sin a(l/r) = (1 + e \cos a)\{\sin (\theta - a) + e \sin \theta\}$$

Show that the locus of the intersection of the normals at the ends of any focal chord of a conic is another conic.

72. From the following theorem derive a second theorem by reciprocation and a third by projective generalisation.

Two parabolas have a common focus (and axes along different lines). The locus of the point of intersection of two perpendicular tangents, one to each of the parabolas, is a conic.

Prove any one of the three theorems

73. If S and S' are two given conics, find the envelope of a line which moves so that the points in which it intersects S harmonically separate the points in which it intersects S'.

Prove that, if the polar with respect to S' of any one point of S, other than a point of intersection of S and S', is divided harmonically by S and S', then the same is true for every point of S.

74. Show that the envelope of a line divided harmonically by two given conics intersecting in four real distinct points is a conic (their "harmonic envelope") touching the tangents to the conics at each of the points

The diagonal points of a quadrangle are A, B, C. Show that the harmonic envelope of a fixed conic having ABC as a self-conjugate triangle, and of a variable conic through the four vertices of the quadrangle touches four fixed lines.

75. Prove that the reciprocal of a circle with respect to a point O is a conic having a focus at O

ABC is the triangle formed by the common tangents of each pair of three parabolas with a common focus. Prove that an infinite number of triangles can be drawn such that each side touches one of the parabolas and each vertex lies on a side of ABC.

Show also that all the triangles are similar, and that one such triangle is formed by the tangents at the vertices of the parabolas

76 Show that the reciprocal of a conic with respect to a point on its director circle is a rectangular hyperbola

Prove that chords of a rectangular hyperbola which subtend a right angle at a point O envelope the parabola which has O as focus and the polar of O with respect to the hyperbola as directrix.

77 Prove that the reciprocal of a conic with respect to a focus is a circle.

Two given straight lines meet at a point O Reciprocate with respect to a circle with centre at O the theorem: the polars of a fixed point, with respect to the circles which touch the two given lines, touch one or other of two parabolas whose axes are parallel to the bisectors of the angles between the lines

78. Prove that the reciprocal of a conic with respect to a focus is a circle, and that the reciprocal of the corresponding directrix is the centre of the circle.

p and q are tangents to a circle, r is their chord of contact, and O is any point in the plane. Prove that a conic can be found having O as focus, r as corresponding directrix, having its asymptotes parallel to p and q, and passing through the two points in which p and q meet the radical axis of the circle and the point-circle O.

79. By reciprocation with respect to the circle ABC, prove that among the conics through the points A, B, C there are three which have a focus at the centre O of the circle ABC; and that their directrices corresponding to O meet in pairs at the middle points of BC, CA, AB.

Show also that these three conics are hyperbolas; and that the second focus of any one of these lies on three conics through O having two of A, B, C for foci

80. From any point O on the circumscribed circle of a triangle ABC, perpendiculars are drawn to the sides BC, CA, AB, meeting the circle in D, E, F respectively. Prove that AD, BE and CF are parallel and that the six lines AB, BC, CD, DE, EF, FA touch a conic.

State the theorem obtained by reciprocation with respect to a circle with centre at O.

81. The chord PQ of a parabola passes through the focus S. Prove that the tangents at P and Q intersect at right angles on the directrix.

State the theorem obtained by reciprocating this result with respect to S.

82. If $AB'CA'BC'$ is a hexagon inscribed in a conic, and if Z_1, Z_2, Z_3 are the points $BC' . B'C$, $CA' . C'A$, $AB' . A'B$ respectively, prove that Z_1, Z_2, Z_3 are collinear.

If $X_1X_2X_3$ is a triangle circumscribing both the triangles ABC, $A'B'C'$, so that X_2X_3, X_3X_1, X_1X_2 fall along AA', BB', CC' respectively, and if $Y_1Y_2Y_3$ is the triangle inscribed in ABC and $A'B'C'$, so that $Y_1 \equiv BC . B'C'$, $Y_2 \equiv CA . C'A'$, $Y_3 \equiv AB . A'B'$, prove that $X_1X_2X_3$ and $Y_1Y_2Y_3$ are in perspective.

Prove that the triangles ABC and $A'B'C'$ are each self-polar with respect to the unique conic which reciprocates the triangle $X_1X_2X_3$ into $Y_1Y_2Y_3$.

83. Prove that the equation of any non-singular conic can be expressed in the form

$$2y = ax^2 + 2hxy + by^2.$$

Prove that the reciprocal, with respect to a conic S, of the circle of curvature Ω at a point P of S, is a conic passing through P whose circle of curvature at P is also Ω.

84. Obtain the equation of the polar reciprocal of the conic $pyz + qzx + rxy = 0$ with respect to the conic $(abcfgh)(xyz)^2 = 0$ in the form

$$p^2X^2 + q^2Y^2 + r^2Z^2 - 2qrYZ - 2rpZX - 2pqXY = 0,$$

where

$$X \equiv ax + hy + gz, \quad Y \equiv hx + by + fz, \quad Z \equiv gx + fy + cz.$$

Given a conic k in the plane of a triangle ABC, prove that, in general, there are four conics through A, B and C whose polar reciprocals with respect to k pass through the two points A and B; but that if A, B are conjugate with respect to k there is just one such conic

Show that the reciprocal of a circle with respect to a rectangular hyperbola is also a circle if and only if the circle is concentric with the hyperbola.

85. If A and B are conjugate points with respect to a circle Σ, prove that the circle on AB as diameter cuts Σ orthogonally

Prove that the locus of the foot of the perpendicular from a given point P to its polar with respect to any circle of a coaxal system is the circle of the orthogonal system passing through P.

Reciprocate this result with respect to a circle centre P.

86. Define the term *one-one correspondence*, and state in what way it is special for an *involution*

$ABCD$ is a quadrilateral and the sides BC, AD; CA, BD; AB, CD meet in X, Y, Z respectively. P is any point on BC, and PA, PD meet YZ in L, M respectively Prove that there is in general one position, P_1, of P for which A, D L_1, M_1 are concyclic, and one position, P_2, of P for which B, C, L_2, M_2, are concyclic and that the radical axis of these two circles cuts YZ at the middle point of the segment of YZ cut off by BC and AD.

87. Show that two conics of a confocal system pass through any given point, one being an ellipse, and the other a hyperbola.

Find the hyperbola confocal with $x^2/a^2 + y^2/b^2 = 1$ which passes through $(a \cos \phi. b \sin \phi)$.

Show that the centre of curvature of the ellipse at this point is the pole of the tangent to the ellipse with respect to the hyperbola.

88. Prove that the locus of the poles of a given straight line with respect to a system of confocal conics is a straight line.

s and s' are the two confocal ellipses which are respectively circumscribed and inscribed to a given triangle ABC. Show that the tangents at A, B, C to s meet at the centres of the escribed circles of the triangle, and that the conic s' touches the three escribed circles.

89. Prove that the polars of a given point P with respect to conics of a confocal system envelope a parabola having directrix OP, where O is the centre of the conics.

If P describes a straight line, prove that the focus of this parabola describes a circle through O.

90. Prove that the polars of a point P, accessible and not on either axis of co-ordinates, with regard to the conics of the confocal system

$$(a + \lambda)l^2 + (b + \lambda)m^2 - n^2 = 0$$

envelope a parabola, which touches the axes of co-ordinates at points Q, R.

Show that when P describes a general line the chord of contact QR turns about a fixed point; and find the locus which P must describe in order that the envelope of QR shall be a circle.

91. In any system of homogeneous co-ordinates the tangential equation of a certain conic is $\Sigma = 0$, that of the absolute points is $\Omega = 0$, while $P = 0$ represents an arbitrary point of the plane. Prove that the envelope of polar lines of P with respect to conics confocal with Σ is the parabola k whose equation is

$$\frac{\partial(\Sigma, P, \Omega)}{\partial(l, m, n)} = 0,$$

and show that this parabola touches the axes of Σ and that its directrix joins the centre of Σ to P.

Prove that k is likewise the envelope of polars of P with respect to conics confocal with any member of the pencil of conics touching Σ at its points of contact with its tangents from P, and that k is the envelope of axes of conics of this pencil.

92 Show that the feet of the normals from a point P to a conic, whose centre is O, lie on a hyperbola through P and O having its asymptotes parallel to the axes of the conic.

A given line meets any conic of a family of confocal central conics in H and K. Show that the line joining the feet of the other two normals to the conic from the intersection of the normals at H and K envelopes a parabola touching the axes of the conic, and show that the chord of contact is perpendicular to the given line.

93. Show that the locus of the centre of the rectangular hyperbola which passes through the intersections of a given conic and any member of a confocal family of conics is a straight line when the confocal conics are a system of confocal parabolas, and is a circle when they are a system of confocal central conics.

94 The plane tangential equations of the vertices of a triangle are $\alpha = 0$, $\beta = 0$, $\gamma = 0$. Interpret the equation $\alpha\beta + \gamma^2 = 0$.

U, V are two opposite intersections of the four common tangents of a given circle centre P and a given conic S. Prove that U, V lie on a conic confocal with S, and that the tangents to this conic at U and V are respectively PU and PV.

95. If x, y, z are trilinear point co-ordinates, prove that there exists a conic inscribed in the triangle of reference which is confocal with the conic $fyz + gzx + hxy = 0$ if, and only if, $f^2 = g^2 = h^2$.

Show that there are, in general, four pairs of confocal conics such that one of them is inscribed and the other circumscribed to the triangle of reference, and that the centres of the four pairs of conics are at the points whose co-ordinates are $(s - a_0, s - b_0, s - c_0)$, $(s, s - c_0, s - b_0)$, $(s - b_0, s - a_0, s)$, where a_0, b_0, c_0 are the sides of the triangle of reference and $2s = a_0 + b_0 + c_0$.

96. Prove that if two triangles are each self-polar with regard to a conic, their vertices lie on a second conic, and conversely.

A triangle ABC is inscribed in a conic k and circumscribed about a conic k'; prove that k and k' are polar reciprocals with regard to a conic s for which the triangle ABC is self-polar; and that any point of k is one vertex of a triangle inscribed in k, circumscribed about k' and self-polar for s. Show also that the vertices of the triangle ABC lie on a conic through the vertices of the common self-polar triangle of k, k'.

Taking k, k' to be concentric rectangular hyperbolas, show that k' is the locus of centres of circumcircles of triangles inscribed in k and circumscribed about k'.

97. Prove that, if two conics S, Ω are so related that there is one triangle inscribed in S and circumscribed about Ω, then there is an infinite number of such triangles.

An ellipse has axes of lengths a, b and the minor axis is BOB', where O is the centre. P is the point on $\overrightarrow{OB}$ produced such that $OP = kOB$, and the tangents from P to the ellipse meet the tangent at B' in points Q, R. Prove that, if $b^2(k^2 - 1) = 3a^2$, then the triangle PQR is equilateral, and find the co-ordinates of its circumcentre.

In the case $k = 5$, find the angles of the triangle, inscribed in the circumcircle of the triangle PQR and circumscribed about the given ellipse, when one side of the triangle is the tangent at B to the ellipse.

98. Prove that a circle which is outpolar to a conic k is cut orthogonally by the orthoptic locus of k.

Prove also that, if a circle is inpolar to a parabola, the diameter of the circle which is perpendicular to the axis of the parabola cuts the circle and parabola in harmonic pairs

Hence, or otherwise, show that a circle which is both outpolar and inpolar to a parabola passes through the focus.

99. Prove that any conic k which is outpolar to two conics whose tangential equations are $\Sigma = 0$, $\Sigma' = 0$ is outpolar to every conic of the range $\Sigma + \lambda\Sigma' = 0$, and show that each of the three point-pairs of this range is a conjugate pair with respect to k.

Show that, in general, there are three point-pairs which are conjugate pairs with respect to each of four given coplanar conics.

100. Prove that two conics which have four-point contact at a point A correspond in a plane perspective with centre A and axis the tangent at A.

Two conics k_1, k_2 have four-point contact at A and are met again by a line through A at B_1, B_2 respectively. An intersection C_2 of k_2 with the tangent at B_1 to k_1 is joined to A by a line meeting k_1 again at C_1. Prove that B_2C_1 is a tangent to k_1.

Show that there is a unique conic r which touches k_1 and k_2 at A and has B_1C_2, B_2C_1 for polars of B_2, C_2; and that k_1, k_2 are polar reciprocals with respect to r

101. Prove that, if a plane collineation has two non-collinear involutory pairs of corresponding points, then it is a harmonic perspective.

Show that, in general collineation of the plane into itself, the circles which transform into circles form a coaxal system.

If a system of coaxal circles (c) is transformed into itself by a plane collineation ω in which neither of the two absolute points is transformed into an absolute point, prove that ω is either (i) a harmonic perspective with one limiting point as vertex and a line through the other as axis, or (ii) one or other of two collineations whose squares are each the reflection in the join of the limiting points of (c).

102. Prove that a plane collineation which transforms the vertices A, B, C, D of a quadrangle into B, A, D, C respectively must be a harmonic perspective.

A collineation ω, in which none of the points A, B, C, D is self-corresponding, transforms two conics k_1, k_2 of the pencil through A, B, C, D into conics of the same pencil. Prove that either ω or ω^2 is a harmonic perspective in which each of k_1, k_2 is self-corresponding.

103. The diagonal points of a quadrangle $ABCD$ are $E \equiv AD \,.\, BC$, $F \equiv BD \,.\, CA$, $G \equiv CD \,.\, AB$; and ω is the plane collineation which transforms A, C, E, G into B, D, G, E respectively. Prove that ω also transforms B, D into C, A and that ω^2 is the harmonic perspective with centre F and axis EG.

Show that every conic through A, B, C, D is invariant in ω^2, but that only one such conic is invariant in ω, other than the line-pair AC, BD.

104. Two chords AB, CD of a conic k meet at O. Prove that there are four plane collineations each of which leaves O and k invariant and transforms the line AB into CD. Show that two of these collineations are harmonic perspectives whose axes meet at O; and that if ω_1, ω_2 are the other two, then $\omega_2\omega_1^{-1}$ is a harmonic perspective with centre O

105. Show that a collineation can be found to transform a conic and any point into a circle and its centre

Two conics S_1 and S_2 meet in A, B, C, D. A chord PQ of S_2 meets the line CD in H, and the harmonic conjugate of H with respect to P and Q lies on S_1. Show that the envelope of the chord is a conic having double contact with S_1 and touching the tangents to S_2 at A, B, C, D.

106. For each of the collineations

$$\text{(i)} \quad x' : y' : z' = y + z : z + x : x + y,$$
$$\text{(ii)} \quad x' : y' : z' = y \quad -x + 2y : -2x + 2y + z,$$

write down the equations of the associated line-line correspondence, and find all the united points and united lines

Indicate clearly the geometrical character of each of the collineations.

107. Find the equation of the harmonic locus F of the conics

$$S \equiv ax^2 + by^2 + cz^2 = 0 \text{ and } S' \equiv x^2 + y^2 + z^2 = 0,$$

and hence infer the tangential equation of the harmonic envelope Φ of S and S'

Verify that the point equation of Φ can be written in the form

$$\Theta' S + \Theta S' - F = 0,$$

where Θ, Θ' are the usual relative invariants of S, S'.

Hence show that for the conics F and Φ to coincide it is necessary and sufficient that triangles can be inscribed in S self-polar with respect to S', and vice versa.

108. If at least three of the common points of two conics are distinct, prove that their equations may be taken as

$$yz + zx + xy = 0, \quad fyz + gzx + hxy = 0.$$

Find the condition that the conics may touch each other, in terms of f, g, h. Express this condition also in terms of the invariants $\Delta, \Theta, \Theta', \Delta'$

109. If two conics passing through three points A, B, C are such that the tangents at A are separated harmonically by AB and AC, prove that

$$\Theta\Theta' = \Delta\Delta'.$$

If the two rectangular hyperbolas

$$2xy = k, \; 2xy + 2gx + 2fy + c = 0$$

are connected by the invariant relation $\Theta\Theta' = \Delta\Delta'$, prove that either (i) the tangents at the points of intersection are equally inclined to the axes, or (ii) $c = k + fg$.

110. Prove that the envelope of lines joining points which correspond in a symmetrical (2, 2) correspondence on a conic is a second conic

A conic k in the plane of a pencil of conics (s) passes through just one base point A of (s) and does not touch all members of (s) at A. Prove that, if P, Q are points of k, other than A, on the same member of (s), then PQ is a tangent to a fixed conic t, which touches the line joining any two base points of (s) different from A.

Determine the tangents from A to t in the cases when (i) the pencil (s) has four base points, (ii) the conics (s) have contact at A.

111. If there is a symmetrical (2, 2) correspondence between points P, P' of a conic S, prove that P, P' are conjugate with respect to a fixed conic Φ and that PP' touches another conic Σ.

Prove that S admits of inscribed quadrilaterals which are circumscribed to Σ if and only if Φ is a line-pair, and show that in this case Φ is a pair of tangents to Σ whose intersection has the same polar for Σ as for S.

112. The co-ordinates of a point P on a curve are given by

$$x : y : z = f(t) : g(t) : h(t)$$

where $f(t)$, $g(t)$, $h(t)$ are polynomials of degree not exceeding n, one of them at least being of degree n. Show that, in general, the curve is of order n and possesses $\frac{1}{2}(n-1)(n-2)$ double points.

Indicate the type of modification to be made in this theorem when $f(t)$, $g(t)$, $h(t)$ can be expressed as polynomials in $k(t)$, where $k(t)$ is itself a polynomial, by considering the curve $x = \{k(t)\}^2$, $y = 2k(t)$ where $k(t) \equiv t^2 + t + 1$.

113. Show that the inflexions of the curve

$$x : y : z = f(t) . \phi(t) : \psi(t)$$

are given by the simple roots of the equation

$$\begin{vmatrix} f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \\ f''(t) & \phi''(t) & \psi''(t) \end{vmatrix} = 0.$$

Find the inflexions and double points of the curve

$$x . y : z = (t^4 + t^3 + t^2) . (t^2 + t + 1) : t^2.$$

114. Find the real inflexion and double point of the curve

$$x . y : z = t^3 + t^2 - 1 \cdot t^3 - t^2 + 1 . t^3 - t.$$

115. Show that a cubic curve drawn through eight fixed points in a plane will also pass through a ninth fixed point

A cubic touches the line $y + z = 0$, $z + x = 0$, $x + y = 0$, at the vertices of the triangle of reference, and the line $x + y + z = 0$ at the point (α, β, γ). Show that it intersects the last line again in the fixed point

$$(\beta^2 - \gamma^2, \quad \gamma^2 - \alpha^2, \quad \alpha^2 - \beta^2)$$

116. Show that the equation of any cuspidal cubic can, by a suitable choice of homogeneous co-ordinates, be written in the form $y^2z = x^3$.

Through B, the inflexion of a cuspidal cubic, are drawn the chords PBP', QBQ'. Any conic through P, P', Q, Q', meets the cubic again at R, R'. Prove that RR' passes through B, and deduce another theorem about the cuspidal cubic by reciprocating this result

117. Prove that two tangents can be drawn from each node of a trinodal quartic to touch the curve elsewhere and that these six tangents touch a conic.

Show also that the three lines, each of which joins the points of contact of the tangents from a node, form a triangle coplanar and coaxal with the triangle formed by the nodes (i.e., is in perspective with it).

118. Show that the tangential equation of a trinodal quartic can be expressed in the form

$$(\Theta\Theta' - 9\Delta\Delta')^2 = 4(\Theta^2 - 3\Delta\Theta')(\Theta'^2 - 3\Delta'\Theta)$$

where Δ, Θ, Θ', Δ' are the invariants of the two conics

$$\begin{aligned} S &\equiv 2(lyz + mzx + nxy) \\ S' &\equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy, \end{aligned}$$

the point equation of the quartic being

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz(fx + gy + hz) = 0.$$

Show that the six inflexional tangents, the six nodal tangents, and the six tangents from the nodes all touch the class quartic

$$\Theta(\Theta'^2 - 4\Delta'\Theta) + 3\Delta\Delta'\Theta' = 0.$$

119. A triangle ABC is inscribed in a conic s, and P is a variable point such that the harmonic conjugate of PA with respect to PB, PC is a tangent to s. Prove that the locus of P is a quartic curve having cusps at A, B, C with concurrent tangents, and touching s at a fourth point Q.

Show also that the tangents to s at the three points (Q) determined in this manner when A, B, C are permuted cyclically form a triangle in perspective with the triangle ABC.

120. Show that the inverse of the circular cubic $xy + (x^2 + y^2)(ax + by) = 0$ with respect to the circle $x^2 + y^2 = 1$ is a certain rectangular hyperbola passing through the origin O.

The two circles, which osculate at O the two branches of the cubic curve, meet again at C; the line OC meets the curve again at P; and the tangent at P meets the curve again at Q. Deduce from the properties of the rectangular hyperbola that (i) P bisects OC, (ii) the bisectors of the angles between OQ and the real asymptote of the cubic curve are parallel to the bisectors of the angles between the two tangents at O.

121. The diagonals AC and BD of a quadrangle $ABCD$ intersect at right angles at O. The perpendiculars from O to AB, BC, CD, DA meet these sides at P, Q, R, S respectively, and the opposite sides at P', Q', R', S'. Prove, by inversion or otherwise, that

(i) P, Q, R, S lie on a circle Σ;
(ii) P' also lies on Σ;
(iii) $P'Q'R'S'$ is a rectangle with its sides parallel to AC and BD.

122. A and B are inverse points with respect to a circle K, and A', B', K' are the inverses of A, B, K with respect to any other circle. Prove that A', B' are inverse with respect to K'.

Two circles K_1, K_2 are exterior to one another, A is any point outside both, and A_1, A_2 are its inverses with respect to K_1, K_2 respectively. Prove that four real circles can be drawn through A to touch both K_1 and K_2, that two of their six intersections (other than A) lie on the circle AA_1A_2, and that the remaining four intersections lie on the circle through A_1, A_2 orthogonal to the circle AA_1A_2.

123. Two non-intersecting circles, centres A and B, each touch internally a circle centre O inside which they lie, and a circle centre C, lying entirely inside the circle centre O, touches each of the first two circles externally. The point O lies outside each of the three circles centres A, B, C. Two circles S_1, S_2 are drawn through O; the first cuts the circles centres A, C orthogonally and the second cuts the circles centres B, C orthogonally. By inversion with respect to the circle centre O, or otherwise, prove that a circle can be drawn to touch OA, OB and also to touch externally each of the circles S_1, S_2.

124. Discuss the curve obtained from the conic $f(x, y, z) = 0$ by the transformation $x = 1/\xi$, $y = 1/\eta$, $z = 1/\zeta$.

Show that this transformation changes a cubic curve passing through two vertices of the triangle of reference and having a node at the third vertex into a conic through the third vertex.

125. A and B are two given points; P is a point which moves so that the pencils PA, PB are in a general (p, q) correspondence. Find the Plücker's numbers n, m, δ, κ, τ, ι of the locus of P.

Find the corresponding numbers when the line AB corresponds to itself in the two pencils.

SYMBOLS

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